

Some topics related to metrics and norms, including ultrametrics and ultranorms, 2

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Abstract

Here we look at (collections of) semimetrics and seminorms, including their ultrametric versions. In particular, we are concerned with geometric properties related to connectedness and topological dimension 0.

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Part I

Semimetrics and seminorms

1 Semimetrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. We say that $d(x, y)$ is a *semimetric* on X if

$$(1.1) \quad d(x, x) = 0$$

for every $x \in X$,

$$(1.2) \quad d(x, y) = d(y, x)$$

for every $x, y \in X$, and

$$(1.3) \quad d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in X$. Of course, (1.3) is known as the *triangle inequality*. If we also have that

$$(1.4) \quad d(x, y) > 0$$

for every $x, y \in X$ with $x \neq y$, then $d(x, y)$ is said to be a *metric* on X .

Let $d(\cdot, \cdot)$ be a semimetric on X , let x be an element of X , and let r be a positive real number. The *open ball* in X centered at x with radius r associated to $d(\cdot, \cdot)$ is defined as usual by

$$(1.5) \quad B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

If $y \in B(x, r)$, then $t = r - d(x, y) > 0$, and it is easy to see that

$$(1.6) \quad B(y, t) \subseteq B(x, r),$$

using the triangle inequality.

A set $U \subseteq X$ is said to be an *open set* with respect to the semimetric $d(\cdot, \cdot)$ if for each $x \in U$ there is an $r > 0$ such that

$$(1.7) \quad B(x, r) \subseteq U.$$

It is well known and easy to check that this defines a topology on X . Note that open balls in X with respect to $d(\cdot, \cdot)$ are open sets, by (1.6). The collection of open balls with respect to $d(\cdot, \cdot)$ centered at a point $x \in X$ is a local base for the topology of X determined by $d(\cdot, \cdot)$ at x , and the collection of all open balls in X with respect to $d(\cdot, \cdot)$ is a base for the topology on X determined by $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a metric on X , then X is Hausdorff with respect to the corresponding topology.

Similarly, the *closed ball* in X centered at a point $x \in X$ with radius $r \geq 0$ with respect to a semimetric $d(\cdot, \cdot)$ is defined by

$$(1.8) \quad \overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

Put

$$(1.9) \quad V(x, r) = X \setminus \overline{B}(x, r) = \{y \in X : d(x, y) > r\}.$$

If $y \in V(x, r)$, then $t = d(x, y) - r > 0$, and one can check that

$$(1.10) \quad B(y, t) \subseteq V(x, r),$$

using the triangle inequality. This implies that $V(x, r)$ is an open set with respect to the topology determined by $d(\cdot, \cdot)$ for every $x \in X$ and $r \geq 0$, so that $\overline{B}(x, r)$ is a closed set with respect to this topology.

Let X be an arbitrary topological space for the moment. Strictly speaking, one often says that X is *regular* if for each $x \in X$ and closed set $E \subseteq X$ with

$x \notin E$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$. This is equivalent to asking that for each $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and the closure \overline{U} of U in X is contained in W . If the topology on X is determined by a semimetric, then it is easy to see that X is regular in this sense, by standard arguments. In particular, a regular topological space in this sense need not be Hausdorff.

If X satisfies the first separation condition, then subsets of X with exactly one element are closed sets, and so regularity of X as in the preceding paragraph implies that X is Hausdorff. In fact, it would be enough to ask that X satisfy the 0th separation separation condition for this to work. Sometimes this may be included in the definition of regularity, and otherwise one may say that a topological space X satisfies the third separation condition when X satisfies the first or 0th separation condition and X is regular.

A topological space X is *normal* in the strict sense if for every pair A, B of disjoint closed subsets of X there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. If X satisfies the first separation condition and is normal in this strict sense, then X is Hausdorff and regular. As before, the first separation condition can also be included in the definition of normality, or one may say that X satisfies the fourth separation condition when X satisfies the first separation condition and X is normal. If the topology on X is determined by a semimetric, then one can check that X is normal in the strict sense, in the same way as for metric spaces.

If $d(x, y)$ is a semimetric on X and $Y \subseteq X$, then the restriction of $d(x, y)$ to $x, y \in Y$ is a semimetric on Y . Let $B_{d,Y}(x, r)$ be the open ball in Y centered at $x \in Y$ with radius $r > 0$ with respect to the restriction of $d(\cdot, \cdot)$ to Y , so that

$$(1.11) \quad B_{d,Y}(x, r) = B_{d,X}(x, r) \cap Y.$$

Thus (1.11) is an open set in Y with respect to the topology induced on Y by the topology on X determined by $d(\cdot, \cdot)$. Every open set in Y with respect to the topology determined by the restriction of $d(\cdot, \cdot)$ to Y can be expressed as a union of open balls in Y , and hence is an open set with respect to the topology induced on Y by the topology on X determined by $d(\cdot, \cdot)$. Conversely, it is easy to see that every open set in Y with respect to the topology induced on Y by the topology on X determined by $d(\cdot, \cdot)$ is an open set with respect to the topology on Y determined by the restriction of $d(\cdot, \cdot)$ to Y , directly from the definitions.

2 Collections of semimetrics

Let X be a set, and let l be a positive integer. If $d_j(x, y)$ is a semimetric on X for $j = 1, \dots, l$, then it is easy to see that

$$(2.1) \quad d(x, y) = \max_{1 \leq j \leq l} d_j(x, y)$$

is a semimetric on X as well. Observe that

$$(2.2) \quad B_d(x, r) = \bigcap_{j=1}^l B_{d_j}(x, r)$$

for every $x \in X$ and $r > 0$, where $B_d(x, r)$ and $B_{d_j}(x, r)$ are as in (1.5). Similarly,

$$(2.3) \quad \tilde{d}(x, y) = \sum_{j=1}^l d_j(x, y)$$

is a semimetric on X , and

$$(2.4) \quad d(x, y) \leq \tilde{d}(x, y) \leq l d(x, y)$$

for every $x, y \in X$. This implies that $d(x, y)$ and $\tilde{d}(x, y)$ determine the same topology on X .

Now let \mathcal{M} be a nonempty collection of semimetrics on X . Let us say that a set $U \subseteq X$ is an *open set* with respect to \mathcal{M} if for each $x \in U$ there are finitely many elements d_1, \dots, d_l of \mathcal{M} and finitely many positive real numbers r_1, \dots, r_l such that

$$(2.5) \quad \bigcap_{j=1}^l B_{d_j}(x, r_j) \subseteq U.$$

It is easy to see that this defines a topology on X . If $\mathcal{M} = \emptyset$, then we interpret the corresponding topology on X as being the indiscrete topology. If \mathcal{M} consists of a single semimetric on X , then this is the same as the topology determined on X by that semimetric, as in the previous section. If \mathcal{M} consists of finitely many semimetrics on X , then the topology on X associated to \mathcal{M} is the same as the topology determined on X by the maximum or sum of the elements of \mathcal{M} . This follows from the remarks in the preceding paragraph, since one can take the r_j 's in (2.5) to be equal to each other. If \mathcal{M} is any collection of semimetrics on X , then every open set in X with respect to the topology determined by any $d \in \mathcal{M}$ is also an open set in X with respect to the topology associated to \mathcal{M} . In particular, open balls in X with respect to the elements of \mathcal{M} are open sets with respect to the topology associated to \mathcal{M} .

Let \mathcal{M} be a nonempty collection of semimetrics on X again, and let $x \in X$ be given, along with finitely many elements d_1, \dots, d_l of \mathcal{M} and positive real numbers r_1, \dots, r_l . Thus $B_{d_j}(x, r_j)$ is an open set with respect to d_j for $j = 1, \dots, l$, and hence with respect to the topology associated to \mathcal{M} , as before. This implies that

$$(2.6) \quad \bigcap_{j=1}^l B_{d_j}(x, r_j)$$

is an open set in X with respect to the topology associated to \mathcal{M} too. If $x \in X$ is fixed, then the collection of open sets of the form (2.6) is a local base for the

topology on X associated to \mathcal{M} at x . The collection of open sets of the form (2.6) for any $x \in X$ is a base for the topology on X associated to \mathcal{M} .

Equivalently, the collection of open balls $B_d(x, r)$ with $d \in \mathcal{M}$ and $r > 0$ is a local sub-base for the topology on X associated to \mathcal{M} at x . Similarly, the collection of open balls $B_d(x, r)$ with $x \in X$, $d \in \mathcal{M}$, and $r > 0$ is a sub-base for the topology on X associated to \mathcal{M} . This is the same as saying that the collection of finite intersections of open balls in X corresponding to elements of \mathcal{M} is a base for the topology on X associated to \mathcal{M} . This is slightly less precise than using intersections of the form (2.6), where the balls are centered at the same point in X .

Every closed set in X with respect to any $d \in \mathcal{M}$ is a closed set in X with respect to the topology on X associated to \mathcal{M} , because of the analogous statement for open sets. This includes closed balls in X with respect to any $d \in \mathcal{M}$, as in the previous section. It follows that the intersection of any family of closed balls in X with respect to elements of \mathcal{M} is a closed set in X with respect to the topology associated to \mathcal{M} as well. Using this, one can check that X is regular in the strict sense discussed in the previous section, with respect to the topology associated to \mathcal{M} .

Let us say that \mathcal{M} is *nondegenerate* on X if for each pair of distinct elements x, y of X there is a $d \in \mathcal{M}$ such that

$$(2.7) \quad d(x, y) > 0.$$

This implies that X is Hausdorff with respect to the topology associated to \mathcal{M} , by essentially the same argument as for metric spaces. If \mathcal{M} consists of finitely many semimetrics on X , then \mathcal{M} is nondegenerate exactly when the sum or maximum of these semimetrics is a metric on X .

If \mathcal{M} is any collection of semimetrics on X and $Y \subseteq X$, then let

$$(2.8) \quad \mathcal{M}_Y$$

be the collection of semimetrics on Y obtained by restricting the elements of \mathcal{M} to Y . One can check that the topology on Y associated to \mathcal{M}_Y as before is the same as the topology induced on Y by the topology on X associated to \mathcal{M} . Of course, this is trivial when $\mathcal{M} = \emptyset$, and so we suppose that $\mathcal{M} \neq \emptyset$. The argument is analogous to the one for a single semimetric, as in the previous section. More precisely, if $E \subseteq Y$ is an open set with respect to the topology induced on Y by the topology on X associated to \mathcal{M} , then it is easy to see that E is an open set with respect to the topology on Y associated to \mathcal{M}_Y , directly from the definitions. Conversely, if $E \subseteq Y$ is an open set with respect to the topology associated to \mathcal{M}_Y , then one would like to verify that E is an open set with respect to the topology induced on Y by the topology on X associated to \mathcal{M} . If E is an open ball in Y centered at a point in Y with respect to an element of \mathcal{M}_Y , then this follows from (1.11). Similarly, if E is the intersection of finitely many open balls in Y with respect to elements of \mathcal{M}_Y , then E can be expressed as the intersection of Y with finitely many open balls in X with respect to elements of \mathcal{M} , which implies that E is an open set in Y with respect

to the topology induced on Y by the topology on X associated to \mathcal{M} . As before, the collection of finite intersection of open balls in Y with respect to elements of \mathcal{M}_Y is a base for the topology on Y associated to \mathcal{M}_Y , so that every open set E in Y with respect to the topology associated to \mathcal{M}_Y can be expressed as a union of finite intersections of open balls in Y with respect to elements of \mathcal{M}_Y . It follows that E is an open set with respect to the topology induced on Y by the topology on X associated to \mathcal{M} , as desired, because E can be expressed as a union of open sets with respect to the induced topology.

3 Seminorms

Let V be a vector space over either the real numbers \mathbf{R} or the complex numbers \mathbf{C} . A nonnegative real-valued function N on V is said to be a *seminorm* on V if it satisfies the following two conditions. First,

$$(3.1) \quad N(tv) = |t|N(v)$$

for every $v \in V$ and real or complex number t , as appropriate, where $|t|$ is the usual absolute value of t . Second,

$$(3.2) \quad N(v+w) \leq N(v) + N(w)$$

for every $v, w \in V$, which is another version of the triangle inequality. Note that (3.1) implies that $N(0) = 0$, by taking $t = 0$. If we also have that

$$(3.3) \quad N(v) > 0$$

for every $v \in V$ with $v \neq 0$, then N is said to be a *norm* on V . If N is a seminorm on V , then it is easy to see that

$$(3.4) \quad d(v, w) = N(v - w)$$

defines a semimetric on V . Similarly, if N is a norm on V , then (3.4) defines a metric on V .

If N_1, \dots, N_l are finitely many seminorms on V , then it is easy to see that

$$(3.5) \quad N(v) = \max_{1 \leq j \leq n} N_j(v)$$

and

$$(3.6) \quad \tilde{N}(v) = \sum_{j=1}^l N_j(v)$$

are seminorms on V too. Let $d(v, w)$ be the semimetric on V corresponding to N as in (3.4), let

$$(3.7) \quad d_j(v, w) = N_j(v - w)$$

be the semimetric corresponding to N_j for each $j = 1, \dots, l$, and let

$$(3.8) \quad \tilde{d}(v, w) = \tilde{N}(v - w)$$

be the semimetric corresponding to \tilde{N} . Thus $d(v, w)$ and $\tilde{d}(v, w)$ can be given in terms of the $d_j(v, w)$'s as in (2.1) and (2.3). As before, we also have that

$$(3.9) \quad N(v) \leq \tilde{N}(v) \leq l N(v)$$

for every $v \in V$.

Now let \mathcal{N} be a collection of seminorms on V , and let \mathcal{M} be the corresponding collection of semimetrics on V , as in (3.4). This leads to a topology on V , as in the previous section. If \mathcal{N} consists of finitely many seminorms on V , then one could get the same topology on V using a single seminorm, as in (3.5) or (3.6). Let us say that \mathcal{N} is *nondegenerate* on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that (3.3) holds. This implies that \mathcal{M} is nondegenerate as a collection of semimetrics on V , as in the previous section, and hence that the associated topology on V is Hausdorff.

If W is a linear subspace of V and N is a seminorm on V , then the restriction of N to W defines a seminorm on W too. Let \mathcal{N} be a collection of seminorms on V again, and let \mathcal{N}_W be the collection of seminorms on W obtained by restricting the elements of \mathcal{N} to W . Also let \mathcal{M} be the collection of semimetrics on V corresponding to \mathcal{N} as before, and let \mathcal{M}_W be the collection of semimetrics on W that correspond to elements of \mathcal{N}_W in the same way. Equivalently, \mathcal{M}_W is the same as the collection of semimetrics on W obtained by restricting the elements of \mathcal{M} to W , as in the previous section. Thus the topology on W associated to \mathcal{N}_W is the same as the topology induced on W by the topology on V associated to \mathcal{N} , as discussed in the previous section again.

4 Semi-ultrametrics

A semimetric $d(x, y)$ on a set X is said to be a *semi-ultrametric* on X if

$$(4.1) \quad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in X$. Note that (4.1) implies the ordinary triangle inequality (1.3). An *ultrametric* on X is a metric that satisfies (4.1). Remember that the *discrete metric* is defined on X by putting $d(x, y)$ equal to 1 when $x \neq y$, and to 0 when $x = y$. It is easy to see that this defines an ultrametric on X , for which the corresponding topology is the discrete topology. One can check that the maximum of finitely many semi-ultrametrics on X is also a semi-ultrametric on X , as in Section 2. However, the sum of finitely many semi-ultrametrics on X is not necessarily a semi-ultrametric on X .

Let $d(\cdot, \cdot)$ be a semi-ultrametric on a set X , and let $x \in X$ and $r > 0$ be given. If $y \in B(x, r)$, then it is easy to see that

$$(4.2) \quad B(y, r) \subseteq B(x, r),$$

using the ultrametric version of the triangle inequality (4.1). More precisely, if $d(x, y) < r$, then

$$(4.3) \quad B(x, r) = B(y, r),$$

since both (4.2) and the opposite inclusion hold, for the same reasons. Similarly, if $y \in \overline{B}(x, r)$ for some $r \geq 0$, then

$$(4.4) \quad \overline{B}(y, r) \subseteq \overline{B}(x, r),$$

by the ultrametric version of the triangle inequality. As before, we get that

$$(4.5) \quad \overline{B}(x, r) = \overline{B}(y, r)$$

when $d(x, y) \leq r$ for some $r \geq 0$, since both (4.4) and the opposite inclusion hold. Note that (4.4) implies that $\overline{B}(x, r)$ is an open set when $r > 0$, with respect to the topology determined by $d(\cdot, \cdot)$. One can also check that $B(x, r)$ is a closed set in X for every $x \in X$ and $r > 0$, and we shall return to this in a moment.

If $x, y, z \in X$ satisfy $d(y, z) \leq d(x, y)$, then (4.1) implies that

$$(4.6) \quad d(x, z) \leq d(x, y).$$

Of course, we also have that

$$(4.7) \quad d(x, y) \leq \max(d(x, z), d(z, y)),$$

by interchanging the roles of y and z in (4.1). If

$$(4.8) \quad d(y, z) < d(x, y),$$

then (4.7) implies that

$$(4.9) \quad d(x, y) \leq d(x, z).$$

Combining (4.6) and (4.9), we get that

$$(4.10) \quad d(x, y) = d(x, z)$$

when $x, y, z \in X$ satisfy (4.8).

Let $x \in X$ and $r \geq 0$ be given, and let $V(x, r)$ be as in (1.9). If $y \in V(x, r)$, then (1.10) holds with $t = d(x, y)$, by (4.10). Similarly, put

$$(4.11) \quad W(x, r) = X \setminus B(x, r) = \{y \in X : d(x, y) \geq r\}$$

for each $r > 0$. If $y \in W(x, r)$, then it is easy to see that

$$(4.12) \quad B(y, d(x, y)) \subseteq W(x, r),$$

using (4.10) again. This implies that $W(x, r)$ is an open set in X with respect to the topology determined by $d(\cdot, \cdot)$, so that $B(x, r)$ is a closed set in X .

5 Absolute value functions

Let k be a field. A nonnegative real-valued function $|x|$ defined on k is said to be an *absolute value function* on k if

$$(5.1) \quad |x| = 0 \quad \text{if and only if} \quad x = 0,$$

and

$$(5.2) \quad |xy| = |x||y|,$$

$$(5.3) \quad |x+y| \leq |x| + |y|$$

for every $x, y \in k$. It is well known that the standard absolute value functions on the real and complex numbers satisfy these conditions. If k is any field, then the *trivial absolute value function* is defined by putting $|x| = 1$ for every $x \in k$ with $x \neq 0$ and $|0| = 0$. It is easy to see that this satisfies the requirements of an absolute value function just mentioned.

Let $|\cdot|$ be an absolute value function on a field k . We have already used 0 to refer to the additive identity elements in k or \mathbf{R} in the preceding paragraph, and we shall use 1 to refer to the multiplicative identity elements in k or \mathbf{R} , depending on the context. If 1 is the multiplicative identity element in k , then $1 \neq 0$ in k , by definition of a field, and hence $|1| > 0$. We also have that $1^2 = 1$ in k , so that $|1| = |1^2| = |1|^2$, which implies that

$$(5.4) \quad |1| = 1.$$

Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n , then

$$(5.5) \quad |x|^n = |x^n| = |1| = 1,$$

so that $|x| = 1$.

The additive inverse of $x \in k$ is denoted $-x$, as usual, so that

$$(5.6) \quad (-1)x = -x$$

for every $x \in k$. In particular, $(-1)^2 = 1$, which implies that

$$(5.7) \quad |-1| = 1,$$

as in the previous paragraph. Combining this with (5.6), we get that

$$(5.8) \quad |-x| = |x|$$

for every $x \in k$. It follows that

$$(5.9) \quad d(x, y) = |x - y|$$

defines a metric on k , using (5.8) to get that (5.9) is symmetric in x and y .

If

$$(5.10) \quad |x+y| \leq \max(|x|, |y|)$$

for every $x, y \in k$, then we say that $|\cdot|$ defines an *ultrametric absolute value function* on k . This condition implies the ordinary triangle inequality (5.3), and that the associated metric (5.10) is an ultrametric on k . It is easy to see that the trivial absolute value function on k is an ultrametric absolute value function, which corresponds to the discrete metric on k . It is well known that the *p-adic absolute value function* defines an ultrametric absolute value function on the field \mathbf{Q} of rational numbers for every prime number p , for which the corresponding ultrametric is known as the *p-adic metric*. The field \mathbf{Q}_p of *p-adic numbers* is obtained by completing \mathbf{Q} with respect to the *p-adic metric*, and the *p-adic absolute value function* can be extended to an ultrametric absolute value function on \mathbf{Q}_p in a natural way.

Let $|\cdot|$ be an ultrametric absolute value function on a field k . If $y, z \in k$ satisfy

$$(5.11) \quad |y - z| < |y|,$$

then

$$(5.12) \quad |y| = |z|.$$

This follows from (4.10) in the previous section, with $x = 0$ and $d(\cdot, \cdot)$ as in (5.9). Of course, one can also verify (5.12) more directly in this situation.

Let \mathbf{Z}_+ be the set of positive integers, and let $n \cdot x$ be the sum of n x 's in a field k for each $n \in \mathbf{Z}_+$ and $x \in k$. An absolute value function $|\cdot|$ on k is said to be *archimedean* if the set of nonnegative real numbers of the form $|n \cdot 1|$ with $n \in \mathbf{Z}_+$ has a finite upper bound, and otherwise $|\cdot|$ is said to be *non-archimedean* on k . If $|\cdot|$ is an ultrametric absolute value function on k , then

$$(5.13) \quad |n \cdot 1| \leq 1$$

for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k . One can check that every non-archimedean absolute value function on k satisfies (5.13) for every $n \in \mathbf{Z}_+$, using (5.2). Equivalently, if $|n_0 \cdot 1| > 1$ for some $n_0 \in \mathbf{Z}_+$, then one can get positive integers n such that $|n \cdot 1|$ is arbitrarily large, by taking powers of n_0 . If an absolute value function $|\cdot|$ on k satisfies (5.13) for every $n \in \mathbf{Z}_+$, then $|\cdot|$ is an ultrametric absolute value function on k , as in Lemma 1.5 on p16 of [1], or Theorem 2.2.2 on p28 of [7]. Thus non-archimedean absolute value functions are in fact ultrametric absolute value functions.

If k has positive characteristic, then there are only finitely many elements of k of the form $n \cdot 1$ for some $n \in \mathbf{Z}_+$. This implies that every absolute value function on k is non-archimedean. If k has only finitely many elements, then every $x \in k$ with $x \neq 0$ satisfies $x^n = 1$ for some $n \in \mathbf{Z}_+$. In this case, the only absolute value function on k is the trivial absolute value function.

6 Seminorms, continued

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k . A nonnegative real-valued function N on V is said to be a *seminorm*

on V with respect to $|\cdot|$ on k if

$$(6.1) \quad N(tv) = |t|N(v)$$

for every $v \in V$ and $t \in k$, and

$$(6.2) \quad N(v+w) \leq N(v) + N(w)$$

for every $v, w \in V$. Of course, this is the same as the definition in Section 3 when $k = \mathbf{R}$ or \mathbf{C} , with the standard absolute value function. As before, (6.1) implies that $N(0) = 0$, by taking $t = 0$, and

$$(6.3) \quad d(v, w) = N(v - w)$$

defines a semimetric on V . If

$$(6.4) \quad N(v) > 0$$

for every $v \in V$ with $v \neq 0$, then N is said to be a *norm* on V , in which case (6.3) defines a metric on V .

Suppose for the moment that $|\cdot|$ is an ultrametric absolute value function on k . If N is a nonnegative real-valued function on V that satisfies (6.1) and

$$(6.5) \quad N(v+w) \leq \max(N(v), N(w))$$

for every $v, w \in V$, then N is said to be a *semi-ultranorm* on V . Note that (6.5) implies (6.2), so that a semi-ultranorm on V is a seminorm. If N is a semi-ultranorm on V , then (6.3) is a semi-ultrametric on V . A semi-ultranorm N on V that satisfies (6.4) is an *ultranorm* on V , which corresponds to an ultrametric on V as in (6.3). If $|\cdot|$ is the trivial absolute value function on k , then the *trivial ultranorm* is defined on V is defined on V by putting $N(v) = 1$ when $v \neq 0$ and $N(0) = 0$. It is easy to see that this is an ultranorm on V , for which the corresponding ultrametric is the discrete metric.

If N_1, \dots, N_l are finitely many seminorms on V with respect to $|\cdot|$ on k , then their maximum and sum are seminorms on V too, as in Section 3. The maximum of finitely many semi-ultranorms is a semi-ultranorm as well. A collection \mathcal{N} of seminorms on V leads to a collection \mathcal{M} of semimetrics on V as in (6.3), and \mathcal{M} determines a topology on V as in Section 2. Let us say that \mathcal{N} is *nondegenerate* on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that (6.4) holds, as in Section 3. As before, this implies that \mathcal{M} is a nondegenerate collection of semimetrics on V , so that the associated topology on V is Hausdorff. The restriction of a seminorm N on V to a linear subspace W of V is a seminorm on W , which is a semi-ultranorm on W when N is a semi-ultranorm on V . Thus a collection \mathcal{N} of seminorms on V leads to a collection \mathcal{N}_W of seminorms on W by restriction, for which the corresponding collection \mathcal{M}_W of semimetrics on W is the same as the collection of restrictions to W of the semimetrics on V in the collection \mathcal{M} associated to \mathcal{N} . As in Section 3, the topology on W associated to \mathcal{N}_W is the same as the topology on W induced by the topology on V associated to \mathcal{N} as before.

Suppose that N is a semi-ultranorm on V . If $v, w \in V$ satisfy

$$(6.6) \quad N(v - w) < N(v),$$

then

$$(6.7) \quad N(v) = N(w).$$

This follows from (4.10), where $d(\cdot, \cdot)$ is as in (6.3), and with $x = 0$, $y = v$, and $z = w$. This is also analogous to (5.12), and can be verified more directly in this case, as before.

7 q -Semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a q -semimetric on X if $d(x, y)^q$ is a semimetric on X . Equivalently, this means that $d(x, y)$ satisfies (1.1), (1.2), and

$$(7.1) \quad d(x, z)^q \leq d(x, y)^q + d(y, z)^q$$

for every $x, y, z \in X$, instead of the usual triangle inequality (1.3). Similarly, if $d(x, y)^q$ is a metric on X , then $d(x, y)$ is said to be a q -metric on X . Note that (7.1) is the same as saying that

$$(7.2) \quad d(x, z) \leq (d(x, y)^q + d(y, z)^q)^{1/q}$$

for every $x, y, z \in X$.

One can define open and closed balls in X with respect to a q -semimetric $d(x, y)$ on X in exactly the same way as for an ordinary semimetric, as in (1.5) and (1.8). Observe that

$$(7.3) \quad B_d(x, r) = B_{d^q}(x, r^q)$$

for every $x \in X$ and $r > 0$, and that

$$(7.4) \quad \overline{B}_d(x, r) = \overline{B}_{d^q}(x, r^q)$$

for every $x \in X$ and $r \geq 0$. Using these open balls in X with respect to $d(\cdot, \cdot)$, one can define a topology on X associated to $d(\cdot, \cdot)$ in the same way as for ordinary semimetrics. The topology on X associated to $d(\cdot, \cdot)$ is the same as the topology on X associated to $d(\cdot, \cdot)^q$, because of (7.3). This implies that the topology associated to a q -semimetric has the same properties as the topology associated to an ordinary semimetric. In particular, open balls with respect to $d(\cdot, \cdot)$ are open sets with respect to the topology on X associated to $d(\cdot, \cdot)$, because of (7.3) and the analogous statement for $d(\cdot, \cdot)^q$. Similarly, closed balls with respect to $d(\cdot, \cdot)$ are closed sets with respect to the topology on X associated to $d(\cdot, \cdot)$, because of (7.4) and the analogous statement for $d(\cdot, \cdot)$. If $Y \subseteq X$, then the restriction of $d(x, y)$ to $x, y \in Y$ defines a q -semimetric on Y , with the same types of properties as in the case of ordinary semimetrics.

Let \mathcal{M} be a collection of q -semimetrics on X , and let $\widetilde{\mathcal{M}}$ be the corresponding collection of ordinary semimetrics on X obtained by taking the q th powers of the elements of \mathcal{M} . More precisely, one can allow $q > 0$ to depend on the element of \mathcal{M} . One can define a topology on X associated to \mathcal{M} in the same way as in Section 2. This is the same as the topology on X associated to $\widetilde{\mathcal{M}}$ as before, because of (7.3), and so the topology on X associated to \mathcal{M} has the same types of properties as before. One can also define nondegeneracy of $\widetilde{\mathcal{M}}$ on X in the same way as before, which is equivalent to nondegeneracy of \mathcal{M} on X .

Let a, b be nonnegative real numbers, and observe that

$$(7.5) \quad \max(a, b) \leq (a^q + b^q)^{1/q} \leq 2^{1/q} \max(a, b).$$

If $0 < q \leq r < \infty$, then it follows that

$$(7.6) \quad a^r + b^r \leq \max(a, b)^{r-q} (a^q + b^q) \leq (a^q + b^q)^{(r-q)/q+1} = (a^q + b^q)^{r/q},$$

so that

$$(7.7) \quad (a^r + b^r)^{1/r} \leq (a^q + b^q)^{1/q}.$$

This implies that every r -semimetric on X is also a q -semimetric on X when $0 < q \leq r < \infty$, using (7.2). Similarly, every semi-ultrametric on X is a q -semimetric on X for each $q > 0$, by the first inequality in (7.5). Semi-ultrametrics on X may be considered as q -semimetrics on X with $q = \infty$, since

$$(7.8) \quad \lim_{q \rightarrow \infty} (a^q + b^q)^{1/q} = \max(a, b)$$

for every $a, b \geq 0$, by (7.5).

8 q -Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|x|$ on k is said to be a q -absolute value function on k if it satisfies (5.1), (5.2), and

$$(8.1) \quad |x + y|^q \leq |x|^q + |y|^q$$

for every $x, y \in k$, instead of (5.3). Equivalently, $|x|$ is a q -absolute value function on k if and only if $|x|^q$ is an ordinary absolute value function on k . In particular, the previous notion of an absolute value function is the same as a q -absolute value function with $q = 1$. If $|\cdot|$ is a q -absolute value function on k , then

$$(8.2) \quad d(x, y) = |x - y|$$

defines a q -metric on k .

As before, (8.1) is the same as saying that

$$(8.3) \quad |x + y| \leq (|x|^q + |y|^q)^{1/q}$$

for every $x, y \in k$. If $0 < q \leq r < \infty$ and $|\cdot|$ is an r -absolute value function on k , then it is easy to see that $|\cdot|$ is a q -absolute value function on k too, using (7.7)

and the reformulation (8.3) of (8.1). Similarly, an ultrametric absolute value function on k is a q -absolute value function on k for every $q > 0$, because of the first inequality in (7.5). As in the previous section, ultrametric absolute value functions may be considered as q -absolute value functions with $q = \infty$, because of (7.8).

Of course, the archimedean and non-archimedean properties can be defined for q -absolute value functions in the same way as for ordinary absolute value functions, as in Section 5. Equivalently, a q -absolute value function $|\cdot|$ on k is archimedean or non-archimedean exactly when $|\cdot|^q$ has the corresponding property as an ordinary absolute value function on k . If $|\cdot|$ is a non-archimedean q -absolute value function on k for some $q > 0$, so that $|\cdot|^q$ is non-archimedean as an ordinary absolute value function on k , then $|\cdot|^q$ is an ultrametric absolute value function on k , as mentioned in Section 5. This implies that $|\cdot|$ is an ultrametric absolute value function on k as well.

Let q_1, q_2 be positive real numbers, and suppose that $|\cdot|_1, |\cdot|_2$ are q_1, q_2 -absolute value functions on k , respectively. Let us say that $|\cdot|_1, |\cdot|_2$ are *equivalent* on k if there is a positive real number a such that

$$(8.4) \quad |x|_2 = |x|_1^a$$

for every $x \in k$. Of course, this implies that the corresponding q_1, q_2 -metrics on k as in (8.2) satisfy an analogous relation, and hence that they determine the same topology on k . Conversely, if the corresponding q_1, q_2 -metrics on k determine the same topology on k , then $|\cdot|_1, |\cdot|_2$ are equivalent in this sense. This follows from Lemma 3.2 on p20 of [1] or Lemma 3.1.2 on p42 of [7] when $q_1 = q_2 = 1$, and otherwise one can reduce to this case by considering $|\cdot|_1^{q_1}, |\cdot|_2^{q_2}$ instead of $|\cdot|_1, |\cdot|_2$.

A famous theorem of Ostrowski implies that every absolute value function on \mathbf{Q} is either trivial, equivalent to the standard Euclidean absolute value function on \mathbf{Q} , or equivalent to the p -adic absolute value function on \mathbf{Q} for some prime number p . See Theorem 2.1 on p16 of [1], or Theorem 3.1.3 on p44 of [7]. If $|\cdot|$ is a q -absolute value function on \mathbf{Q} for some positive real number q , then the same conclusion holds, since Ostrowski's theorem can be applied to $|\cdot|^q$ as an ordinary absolute value function on \mathbf{Q} .

If k is a field of characteristic 0, then it is well known that there is a natural embedding of \mathbf{Q} into k . If $|\cdot|$ is a q -absolute value function on k for some $q > 0$, then $|\cdot|$ induces a q -absolute value function on \mathbf{Q} , using this embedding. Note that $|\cdot|$ is archimedean or non-archimedean on k exactly when the induced q -absolute value function on \mathbf{Q} has the same property. In particular, if $|\cdot|$ is archimedean on k , then the induced q -absolute value function is archimedean on \mathbf{Q} , and hence is equivalent to the standard Euclidean absolute value function on \mathbf{Q} , by Ostrowski's theorem.

Let k be any field again, and let $|\cdot|$ be a nonnegative real-valued function on k . If $|\cdot|$ satisfies (5.1) and (5.2), then $|\cdot|^a$ has the same properties for every $a > 0$. If $|\cdot|$ is a q -absolute value function on k for some $q > 0$, then $|\cdot|^a$ is a (q/a) -absolute value function on k . Similarly, if $|\cdot|$ is an ultrametric absolute

value function on k , then $|\cdot|^a$ is an ultrametric absolute value function on k for every $a > 0$, which was implicitly mentioned earlier. If $|\cdot|$ is the standard Euclidean absolute value function on \mathbf{Q} , then $|\cdot|^a$ is not an absolute value function on \mathbf{Q} for any $a > 1$, which is the same as saying that $|\cdot|$ is not a q -absolute value function on \mathbf{Q} for any $q > 1$.

9 q -Seminorms

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Also let V be a vector space over k , and let N be a nonnegative real-valued function on V . Let us say that N is a q -seminorm on V with respect to $|\cdot|$ on k if N satisfies the homogeneity condition (6.1), and if

$$(9.1) \quad N(v + w)^q \leq N(v)^q + N(w)^q$$

for every $v, w \in V$. If $q = 1$, then $|\cdot|$ is an ordinary absolute value function on k , and this is the same as saying that N is an ordinary seminorm on V with respect to $|\cdot|$ on k , as in Section 6. If q is any positive real number, then $|\cdot|$ is a q -absolute value function on k if and only if $|\cdot|^q$ is an ordinary absolute value function on k , in which case N is a q -seminorm on V with respect to $|\cdot|$ on k if and only if $N(v)^q$ is an ordinary seminorm on V with respect to $|\cdot|^q$ on k . If N is a q -seminorm on V with respect to $|\cdot|$ on k for any $q > 0$, then

$$(9.2) \quad d(v, w) = N(v - w)$$

defines a q -semimetric on V . If we also have that $N(v) > 0$ for every $v \in V$ with $v \neq 0$, then N is said to be a q -norm on V . In this case, (9.2) defines a q -metric on V .

As usual, (9.1) is the same as saying that

$$(9.3) \quad N(v + w) \leq (N(v)^q + N(w)^q)^{1/q}$$

for every $v, w \in V$. Suppose that $|\cdot|$ is an r -absolute value function on k for some $r > 0$, and that N is an r -seminorm with respect to $|\cdot|$ on k . If $0 < q \leq r$, then $|\cdot|$ is a q -absolute value function on k too, as in the previous section. It is easy to see that N is a q -seminorm on V with respect to $|\cdot|$ on k under these conditions, using (7.7), and the reformulation (9.3) of (9.1). Similarly, if $|\cdot|$ is an ultrametric absolute value function on k , then $|\cdot|$ is a q -absolute value function on k for every $q > 0$. If N is a semi-ultranorm on V with respect to $|\cdot|$ on k , then it is easy to see that N is a q -seminorm on V with respect to $|\cdot|$ on k for every $q > 0$, using the first inequality in (7.5). As before, semi-ultranorms may be considered as q -seminorms with $q = \infty$, because of (7.8).

Let $|\cdot|$ be a q -absolute value function on k for some $q > 0$ again, and hence for some range of q 's, which may include $q = \infty$. Also let \mathcal{N} be a collection of q -seminorms on V with respect to $|\cdot|$ on k , where q is allowed to depend on the element of \mathcal{N} , as long as $|\cdot|$ is a q -absolute value function on k . This leads to a collection \mathcal{M} of semimetrics on V , associated to the elements of \mathcal{N} as in (9.2),

and where q is allowed to depend on the element of \mathcal{M} . Using \mathcal{M} , we can get a topology on V , as in Sections 2 and 7. As before, \mathcal{N} is said to be *nondegenerate* on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$, in which case \mathcal{M} is nondegenerate on V too.

Let $|\cdot|$ be a nonnegative real-valued function on k , let N be a nonnegative real-valued function on V , and suppose that $N(v) > 0$ for some $v \in V$. If N satisfies the homogeneity condition (6.1), then $|\cdot|$ has to satisfy the multiplicative property (5.2). One can also use this to get that $|x| > 0$ for every $x \in k$ with $x \neq 0$. If N satisfies (9.1) for some $q > 0$ too, then the homogeneity condition (6.1) implies that $|\cdot|$ satisfies (8.1) in the previous section. Similarly, if N satisfies the ultrametric version of the triangle inequality (6.5), then the homogeneity condition (6.1) implies that $|\cdot|$ has the analogous property (5.10).

10 Balanced sets

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . If $E \subseteq V$ and $t \in k$, then put

$$(10.1) \quad tE = \{tv : v \in E\}$$

and

$$(10.2) \quad -E = (-1)E = \{-v : v \in E\}.$$

A set $E \subseteq V$ is said to be *balanced* with respect to $|\cdot|$ on k if

$$(10.3) \quad tE \subseteq E$$

for every $t \in k$ with $|t| \leq 1$. In particular, this implies that

$$(10.4) \quad tE = E$$

for every $t \in k$ with $|t| = 1$, since (10.3) holds with t replaced by $1/t$ when $|t| = 1$. Sometimes balanced sets are said to be *circled*, although this could also be used to refer to (10.4), especially when $k = \mathbf{C}$ with the standard absolute value function. Similarly, $E \subseteq V$ is said to be *symmetric* (about the origin) if (10.3) holds with $t = -1$, which is the same as saying that (10.4) holds with $t = -1$. Note that nonempty balanced sets automatically contain 0, by taking $t = 0$ in (10.3). If $|\cdot|$ is the trivial absolute value function on k , then $E \subseteq V$ is balanced if and only if either $E = \emptyset$ or $0 \in E$ and E satisfies (10.4) for every $t \in k$ with $t \neq 0$. If $|\cdot|$ is any q -absolute value function on k , then the union and intersection of any collection of balanced subsets of V are balanced in V too. If E is any subset of V , then

$$(10.5) \quad \bigcup\{tE : t \in k, |t| \leq 1\}$$

is a balanced subset of V that contains E . This is the smallest balanced subset of V that contains E , which may be described as the *balanced hull* of E in V .

Suppose for the moment that $k = \mathbf{R}$ or \mathbf{C} , with the standard absolute value function. A set $E \subseteq V$ is said to be *starlike about 0* if (10.3) holds for every $t \in \mathbf{R}$ with $0 \leq t \leq 1$. As before, this implies that $0 \in E$ when $E \neq \emptyset$, by taking $t = 0$. If $k = \mathbf{R}$, then $E \subseteq V$ is balanced if and only if E is both symmetric and starlike about 0. If $k = \mathbf{C}$, then E is balanced if and only if E is starlike about 0 and E satisfies (10.4) for every $t \in \mathbf{C}$ with $|t| = 1$. In both cases, the union and intersection of any collection of starlike subsets of V about 0 are starlike in V about 0 as well. In particular, for any $E \subseteq V$,

$$(10.6) \quad \bigcup\{tE : t \in \mathbf{R}, 0 \leq t \leq 1\}$$

is starlike about 0 in V and contains E . As before, this is the smallest subset of V that is starlike about 0 and contains E , and which may be described as the *starlike hull* of E in V about 0. If $k = \mathbf{R}$ and E is symmetric about 0, then the balanced hull of E in V is the same as the starlike hull of E in V about 0. Similarly, if $k = \mathbf{C}$ and E satisfies (10.4) for every $t \in \mathbf{C}$ with $|t| = 1$, then the balanced hull of E in V is the same as the starlike hull of E in V about 0.

Let $|\cdot|$ be a q -absolute value function on any field k again, and let N be a nonnegative real-valued function on V that satisfies the usual homogeneity condition (6.1) with respect to $|\cdot|$ on k . Observe that

$$(10.7) \quad B_N(0, r) = \{v \in V : N(v) < r\}$$

is a balanced subset of V for each $r > 0$, and that

$$(10.8) \quad \overline{B}_N(0, r) = \{v \in V : N(v) \leq r\}$$

is balanced for every $r \geq 0$. More precisely,

$$(10.9) \quad t B_N(0, r) = B_N(0, |t|r)$$

for every $r > 0$ and $t \in k$ with $t \neq 0$, and

$$(10.10) \quad t \overline{B}_N(0, r) = \overline{B}_N(0, |t|r)$$

for every $r \geq 0$ and $t \in k$ with $t \neq 0$. If $t = 0$, then (10.10) may not hold, since there may be $v \in V$ with $v \neq 0$ and $N(v) = 0$.

Let us return now to the case where $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, and let N be a nonnegative real-valued function on V . Suppose that N is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, which is to say that

$$(10.11) \quad N(tv) = tN(v)$$

for every $v \in V$ and $t \geq 0$. As before, (10.7) is starlike about 0 in V for every $r > 0$, and (10.8) is starlike about 0 in V for every $r \geq 0$. Similarly, (10.9) holds for every $r, t > 0$, and (10.10) holds for every $r \geq 0$ and $t > 0$, where $|t|$ reduces to t on the right sides of these equations. If $k = \mathbf{R}$, then N satisfies

(6.1) exactly when N satisfies (10.11) and N is invariant under multiplication by -1 . If $k = \mathbf{C}$, then N satisfies (6.1) exactly when N satisfies (10.11) and N is invariant under multiplication by complex numbers with absolute value equal to 1. In both cases, we are back to the situation discussed in the preceding paragraph.

11 Absorbing sets

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$ again, and let V be a vector space over k . A set $A \subseteq V$ is said to be *absorbing* with respect to $|\cdot|$ on k if for each $v \in V$ there is a $t_0(v) \in k$ such that $t_0(v) \neq 0$ and

$$(11.1) \quad t v \in A$$

for every $t \in k$ with $|t| \leq |t_0(v)|$. Equivalently, $A \subseteq V$ is absorbing if for each $v \in V$ there is a $t_1(v) \in k$ such that

$$(11.2) \quad v \in t A$$

for every $t \in k$ with $|t| \geq |t_1(v)|$. More precisely, both versions of the absorbing condition imply that $0 \in A$. If A satisfies the first version of the absorbing condition, then A satisfies the second version of the absorbing condition with $t_1(v) = 1/t_0(v)$. Conversely, if A satisfies the second version of the absorbing condition, then A satisfies the first version of the absorbing condition with $t_0(v) = 1/t_1(v)$ when $t_1(v) \neq 0$. This uses the fact that $0 \in A$ to get that (11.1) holds when $t = 0$. If $t_1(v) = 0$, then (11.2) holds for every $t \in k$, including $t = 0$. This implies that $v = 0$, and that (11.1) holds for every $t \in k$, so that one can take $t_0(v)$ to be any nonzero element of k , such as the multiplicative identity element 1.

An absorbing set $A \subseteq V$ is also said to be *radial* (at 0). This is especially natural when $k = \mathbf{R}$ with the standard absolute value function. In this case, it suffices to consider only nonnegative real numbers t in (11.1) and (11.2), since analogous conditions for negative real numbers may be obtained by considering $-v$ instead of v . If $k = \mathbf{C}$ with the standard absolute value function, then one might consider the absorbing or radial property of a subset A of V as a real vector space, instead of the stronger condition for V as a complex vector space. The two conditions are equivalent in some situations, such as when A is invariant under multiplication by complex numbers with absolute value equal to 1, and when A is convex.

As a weaker version of the absorbing property, one can ask that for each $v \in V$ there be a $t \in k$ such that $t \neq 0$ and (11.1) holds. This is equivalent to asking that $0 \in A$ and for each $v \in V$ there be a $t \in k$ that satisfies (11.2), for the same types of reasons as before. If A is balanced with respect to $|\cdot|$ on k , then this weaker condition implies that A is absorbing. If $k = \mathbf{R}$, then one might refine this weaker condition a bit by restricting one's attention to $t \geq 0$. If A is starlike about 0 and satisfies this refined version of the weaker condition, then A is absorbing with respect to the standard absolute value function on \mathbf{R} .

If $|\cdot|$ is the trivial absolute value function on k , then $A \subseteq V$ is absorbing if and only if $A = V$. Suppose now that $|\cdot|$ is a nontrivial q -absolute value function on k . In this case, the first version of the absorbing property may be reformulated as saying that for each $v \in V$ there is a positive real number $r_0(v)$ such that (11.1) holds for every $t \in k$ with $|t| \leq r_0(v)$. Similarly, the second version of the absorbing condition can be reformulated as saying that for each $v \in V$ there is a nonnegative real number $r_1(v)$ such that (11.2) holds for every $t \in k$ with $|t| \geq r_1(v)$. This uses the fact that $|t|$ can take arbitrarily large and small positive values with $t \in k$ when $|\cdot|$ is nontrivial on k . If N is a nonnegative real-valued function on V that satisfies the homogeneity condition (6.1), then the open and closed balls (10.7) and (10.8) centered at 0 in V with respect to N are absorbing in V for every $r > 0$. If $k = \mathbf{R}$ with the standard absolute value function, and if N is a nonnegative real-valued function on V that is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, as in (10.11), then the corresponding open and closed balls (10.7) and (10.8) are absorbing in V for every $r > 0$ too. If $k = \mathbf{C}$ with the standard absolute value function, then one can apply the previous statement to V as a vector space over \mathbf{R} .

12 Discrete absolute value functions

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Let us say that $|\cdot|$ is *discrete* on k if there is a nonnegative real number $\rho < 1$ such that

$$(12.1) \quad |x| \leq \rho$$

for every $x \in k$ with $|x| < 1$. Equivalently, if $y, z \in k$ satisfy $|y| < |z|$, then

$$(12.2) \quad |y| \leq \rho |z|,$$

by applying (12.1) to $x = y/z$. In particular, if $|z| > 1$, then (12.2) implies that

$$(12.3) \quad |z| \geq 1/\rho > 1,$$

by taking $y = 1$. More precisely, if $\rho = 0$, then $1/\rho$ is interpreted as being $+\infty$, and $|\cdot|$ is the trivial absolute value function on k .

Observe that

$$(12.4) \quad \{|x| : x \in k, x \neq 0\}$$

is a subgroup of the multiplicative group \mathbf{R}_+ of positive real numbers. If $|\cdot|$ is discrete on k , then it is easy to see that (12.4) has no limit points in \mathbf{R}_+ with respect to the standard Euclidean metric on \mathbf{R} , because of (12.2). However, 0 is a limit point of (12.4) with respect to the standard Euclidean metric on \mathbf{R} when $|\cdot|$ is nontrivial on k . Conversely, if 0 is not a limit point of (12.4) with respect to the standard Euclidean metric on \mathbf{R}_+ , then $|\cdot|$ satisfies (12.1) for some $\rho < 1$, and so $|\cdot|$ is discrete on k . If $|\cdot|$ is not discrete on k , then (12.4) is dense in \mathbf{R}_+ with respect to the standard Euclidean metric.

If $|\cdot|$ is archimedean on k , then k has characteristic 0, so that there is a natural embedding of \mathbf{Q} in k . The induced q -absolute value function on \mathbf{Q} is also archimedean under these conditions, which implies that the induced q -absolute value function on \mathbf{Q} is equivalent to the standard Euclidean absolute value function on \mathbf{Q} , by Ostrowski's theorem. Of course, the set of positive values of the standard Euclidean absolute value function on \mathbf{Q} is the same as the set \mathbf{Q}_+ of positive rational numbers, which is dense in \mathbf{R}_+ with respect to the standard Euclidean metric. It follows that (12.4) is dense in \mathbf{R}_+ too, since (12.4) includes the positive values of a q -absolute value function on \mathbf{Q} that is equivalent to the standard Euclidean absolute value function on \mathbf{Q} . Thus a discrete q -absolute value function on a field k has to be non-archimedean, and hence an ultrametric absolute value function.

Let $|\cdot|$ be a nontrivial discrete q -absolute value function on a field k , and put

$$(12.5) \quad \rho_1 = \sup\{|x| : x \in k, |x| < 1\},$$

so that $0 < \rho_1 < 1$. It is easy to see that there is an $x_1 \in k$ such that

$$(12.6) \quad |x_1| = \rho_1,$$

because ρ_1 is an element of the closure of (12.4) in \mathbf{R}_+ , and (12.4) has no limit points in \mathbf{R}_+ , by hypothesis. Of course,

$$(12.7) \quad |x_1^j| = |x_1|^j = \rho_1^j$$

for each integer j , which means that (12.4) contains all integer powers of ρ_1 . Note that (12.1) holds with $\rho = \rho_1$, by construction, so that (12.2) holds with $\rho = \rho_1$ too. Using this, one can check that every element of (12.4) is an integer power of ρ_1 in this situation.

13 Balanced q -convexity

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . Let us say that a balanced set $E \subseteq V$ is q -convex with respect to $|\cdot|$ on k if for every $v_1, v_2 \in E$ and $t_1, t_2 \in k$ with

$$(13.1) \quad |t_1|^q + |t_2|^q \leq 1,$$

we have that

$$(13.2) \quad t_1 v_1 + t_2 v_2 \in E.$$

Suppose that N is a nonnegative real-valued function on V that satisfies the homogeneity condition (6.1) with respect to $|\cdot|$ on k . If $B_N(0, r)$ and $\overline{B}_N(0, r)$ are as in (10.7) and (10.8), respectively, then $B_N(0, r)$ is balanced in V for every $r > 0$, and $\overline{B}_N(0, r)$ is balanced in V for every $r \geq 0$, as before. If N is a q -seminorm on V with respect to $|\cdot|$ on k , then it is easy to see that $B_N(0, r)$ is q -convex for every $r > 0$, and $\overline{B}_N(0, r)$ is q -convex for every $r \geq 0$.

Suppose for the moment that $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, and that $0 < q \leq 1$. Let us say that a starlike set $E \subseteq V$ about 0 is *real q -convex* if for every $v_1, v_2 \in E$ and nonnegative real numbers t_1, t_2 with

$$(13.3) \quad t_1^q + t_2^q \leq 1,$$

we have that (13.2) holds. If $E \subseteq V$ is balanced, then E is starlike about 0 in particular. In this case, real q -convexity is equivalent to q -convexity as defined in the preceding paragraph. Remember that a set $E \subseteq V$ is convex in the ordinary sense if (13.2) holds for every $v_1, v_2 \in E$ and nonnegative real numbers t_1, t_2 such that

$$(13.4) \quad t_1 + t_2 = 1.$$

If $E \subseteq V$ is convex and $0 \in E$, then E is starlike about 0, and real 1-convex. Of course, real 1-convexity implies ordinary convexity.

Let N be a nonnegative real-valued function on V that is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, as in (10.11). Thus $B_N(0, r)$ and $\overline{B}_N(0, r)$ can be defined as in (10.7) and (10.8), respectively, $B_N(0, r)$ is starlike about 0 in V for every $r > 0$, and $\overline{B}_N(0, r)$ is starlike about 0 in V for every $r \geq 0$. Let us say that N is *q -subadditive* on V if

$$(13.5) \quad N(v + w)^q \leq N(v)^q + N(w)^q$$

for every $v, w \in V$. In this case, $B_N(0, r)$ is real q -convex in V for every $r > 0$, and $\overline{B}_N(0, r)$ is real q -convex for every $r \geq 0$. Note that N is a q -seminorm on V when N is also invariant under multiplication by -1 in the real case, and when N is invariant under multiplication by complex numbers with absolute value equal to 1 in the complex case.

Let $|\cdot|$ be a q -absolute value function on a field k for some $q > 0$ again, and observe that (13.1) is equivalent to

$$(13.6) \quad (|t_1|^q + |t_2|^q)^{1/q} \leq 1.$$

If $0 < \tilde{q} \leq q$, then $|\cdot|$ is also a \tilde{q} -absolute value function on k , as in Section 8. Similarly, if $E \subseteq V$ is balanced and q -convex with respect to $|\cdot|$ on k , and if $0 < \tilde{q} \leq q$, then E is \tilde{q} -convex. This uses the fact that the left side of (13.6) is monotonically decreasing in q , as in Section 7. If $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, $E \subseteq V$ is starlike about 0 and real q -convex, and $0 < \tilde{q} \leq q$, then E is real \tilde{q} -convex too, for essentially the same reasons. As usual, (13.5) can be reformulated as saying that

$$(13.7) \quad N(v + w) \leq (N(v)^q + N(w)^q)^{1/q}$$

for every $v, w \in V$. If $0 < \tilde{q} \leq q$, then q -subadditivity implies \tilde{q} -subadditivity, just as for q -seminorms.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on a field k , and let E be a balanced subset of V . The analogue of q -convexity with $q = \infty$

with respect to $|\cdot|$ on k asks that (13.2) hold for every $v_1, v_2 \in E$ and $t_1, t_2 \in k$ with

$$(13.8) \quad \max(|t_1|, |t_2|) \leq 1.$$

This is the same as saying that

$$(13.9) \quad v_1 + v_2 \in E$$

for every $v_1, v_2 \in E$, because E is balanced. If $E \neq \emptyset$, so that $0 \in E$, then this means that E is a subgroup of V with respect to addition, because E is symmetric about the origin. Note that this property implies that E is q -convex for every $q > 0$.

Let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (6.1) with respect to $|\cdot|$ on k again. If N is a semi-ultranorm on V , then $B_N(0, r)$ has the property described in the preceding paragraph for every $r > 0$, and $\overline{B}_N(0, r)$ has this property for every $r \geq 0$. Conversely, if $\overline{B}_N(0, r)$ has this property for every $r \geq 0$, then N satisfies the ultrametric version of the triangle inequality, as in (6.5). Similarly, N satisfies the ultrametric version of the triangle inequality when $B_N(0, r)$ has the property just mentioned for every $r > 0$.

Suppose for the moment that $|\cdot|$ is trivial on k . If $0 < q < \infty$, then (13.1) holds if and only if at least one of t_1 and t_2 is equal to 0. This implies that every balanced set $E \subseteq V$ with respect to $|\cdot|$ on k is q -convex when $0 < q < \infty$. In this case, $E \subseteq V$ is balanced with respect to $|\cdot|$ on k if and only if $tE \subseteq E$ for every $t \in k$. It follows that a balanced set $E \subseteq V$ satisfies the $q = \infty$ version of q -convexity with respect to $|\cdot|$ on k if and only if E is either empty or a linear subspace of V .

Let $|\cdot|$ be an arbitrary q -absolute value function on k again, for some $q > 0$. Note that the properties of being balanced and q -convex with respect to $|\cdot|$ on k are preserved by linear mappings, and by scalar multiplication in particular. Remember that the union and intersection of any collection of balanced subsets of V are also balanced in V , as in Section 10. The intersection of any collection of balanced q -convex subsets of V is q -convex in V too. If a collection of balanced q -convex subsets of V is linearly ordered by inclusion, then the union of these sets is q -convex in V as well. If $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function and $0 < q \leq 1$, then there are analogous statements for starlike subsets of V about 0 and starlike sets that are real q -convex. Of course, this is also analogous to the situation for subsets of V that are convex in the ordinary sense.

14 Minkowski functionals

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$ again, and let V be a vector space over k . Also let A be a balanced absorbing subset of V , and put

$$(14.1) \quad N_A(v) = \inf\{|t| : t \in k, v \in tA\}$$

for each $v \in V$. If $|\cdot|$ is nontrivial on k , then $N_A(v)$ can be defined equivalently by

$$(14.2) \quad \begin{aligned} N_A(v) &= \inf\{|t| : t \in k \setminus \{0\}, v \in t A\} \\ &= \inf\{|t| : t \in k \setminus \{0\}, t^{-1} v \in A\} \end{aligned}$$

for every $v \in V$. Otherwise, (14.2) can only differ from (14.1) when $v = 0$, which is the only case where $t = 0$ can be used in (14.1). If $|\cdot|$ is the trivial absolute value function on k , then we have seen that V is the only absorbing subset of itself. If $|\cdot|$ is the trivial absolute value function on k and $A = V$, then (14.1) is the trivial ultranorm on V , while (14.2) is equal to 1 for every $v \in V$. However, if $|\cdot|$ is nontrivial on k and $A = V$, then both (14.1) and (14.2) are equal to 0 for every $v \in V$.

More precisely, in order to define $N_A(v)$ as in (14.1), it suffices to ask that for each $v \in V$ there be a $t \in k$ such that $v \in t A$, so that the infimum in (14.1) is taken over a nonempty set. Similarly, in order to define $N_A(v)$ as in (14.2), it suffices to ask that for each $v \in V$ there be a $t \in k \setminus \{0\}$ such that $v \in t A$. Of course, the second condition holds if and only if the first condition holds and $0 \in A$. If A has either of these properties, then the balanced hull of A in V is both balanced and absorbing, as in Section 11. In both cases, one can check that (14.1) and (14.2) are the same for A as for the balanced hull of A in V . Thus we may as well ask that A be balanced and absorbing, as in the preceding paragraph. Note that N_A automatically satisfies the homogeneity condition (6.1).

Suppose for the moment that $k = \mathbf{R}$ with the standard absolute value function. Instead of (14.1), it is customary to consider

$$(14.3) \quad \tilde{N}_A(v) = \inf\{t \in \mathbf{R}_+ \cup \{0\} : v \in t A\}$$

for each $v \in V$. As before, one should ask that for each $v \in V$ there be a $t \geq 0$ such that $v \in t A$, so that the infimum in (14.3) is taken over a nonempty set of nonnegative real numbers. Similarly, the analogue of (14.2) in this case is

$$(14.4) \quad \tilde{N}_A(v) = \inf\{t \in \mathbf{R}_+ : v \in t A\} = \inf\{t \in \mathbf{R}_+ : t^{-1} v \in A\}$$

for each $v \in V$. In this case, one should ask that for each $v \in V$ there be a $t \in \mathbf{R}_+$ such that $v \in t A$, so that the infimum is taken over a nonempty set of positive real numbers. As in the previous situation, this condition is equivalent to the corresponding condition for (14.3) together with the additional requirement that $0 \in A$. If this condition holds, then (14.3) is equal to (14.4) for every $v \in V$, for essentially the same reasons as before.

If A is absorbing in V , then A satisfies the conditions needed to define (14.3) and (14.4). In the other direction, if A satisfies the condition needed to define (14.3) for every $v \in V$, and if A is starlike about 0, then A is absorbing in V . More precisely, this uses the fact that one can define the absorbing property for subsets of real vector spaces in terms of nonnegative real numbers, as in Section 11. If A satisfies either of the conditions needed to define (14.3) or (14.4), then

the starlike hull of A in V about 0 is absorbing in V and starlike about 0. As in the previous situation, (14.3) and (14.4) are the same for A as for the starlike hull of A in V about 0, and so one may as well suppose that A is absorbing in V and starlike about 0. It is easy to see that $\tilde{N}_A(v)$ is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers on V , as in (10.11). If A is symmetric about 0, then (14.1) is the same as (14.3), and (14.2) is the same as (14.4). In particular, $\tilde{N}_A(v)$ is invariant under multiplication by -1 on V in this case.

If $k = \mathbf{C}$ with the standard absolute value function, then one can also treat V as a vector space over \mathbf{R} , so that the previous remarks can be applied. In this case, if $A \subseteq V$ is invariant under multiplication by complex numbers with absolute value equal to 1, then (14.1) is the same as (14.3), (14.2) is the same as (14.4), and the conditions under which they are defined are equivalent. It follows that $\tilde{N}_A(v)$ is invariant under multiplication by complex numbers with absolute value equal to 1 too, which could also be verified directly from the definitions. As in Section 11, A is absorbing in V as a complex vector space if and only if A is absorbing in V as a real vector space under these conditions. We also have that A is balanced in V as a complex vector space when A is starlike about 0 in this situation, and that the balanced hull of A in V as a complex vector space is the same as the starlike hull of A in V about 0.

Let $|\cdot|$ be a q -absolute value function on a field k again, and let A be a balanced absorbing subset of V . It is easy to see that

$$(14.5) \quad B_{N_A}(0, 1) \subseteq A \subseteq \overline{B}_{N_A}(0, 1),$$

where $B_{N_A}(0, 1)$ and $\overline{B}_{N_A}(0, 1)$ are as in (10.7) and (10.8). More precisely, the second inclusion in (14.5) simply says that $N_A(v) \leq 1$ when $v \in A$, which holds by the definition (14.1) of $N_A(v)$. To get the first inclusion in (14.5), observe that if $v \in V$ and $N_A(v) < 1$, then there is a $t \in k$ such that $v \in tA$ and $|t| < 1$, by the definition (14.1) of $N_A(v)$. If A is balanced, then this implies that $v \in A$, as desired. Similarly, if $k = \mathbf{R}$ with the standard absolute value function, and $A \subseteq V$ is starlike about 0 and absorbing, then

$$(14.6) \quad B_{\tilde{N}_A}(0, 1) \subseteq A \subseteq \overline{B}_{\tilde{N}_A}(0, 1).$$

If $k = \mathbf{C}$ with the standard absolute value function, then one can also treat V as a real vector space, and apply the previous statement.

Suppose now that $|\cdot|$ is a nontrivial discrete q -absolute value function on a field k . This implies that

$$(14.7) \quad \{|x| : x \in k\}$$

is a closed subset of the real line with respect to the standard topology, and that 0 is the only limit point of (14.7) in \mathbf{R} . If $A \subseteq V$ is balanced and absorbing, as before, then it follows that $N_A(v)$ is an element of (14.7) for every $v \in V$. More precisely, if $N_A(v) > 0$, then the infimum in (14.1) is attained, and so there is a $t \in k$ such that $v \in tA$ and $N_A(v) = |t|$. In this case, we have that

$$(14.8) \quad A = \overline{B}_{N_A}(0, 1),$$

instead of (14.5).

Let $|\cdot|$ be an ultrametric absolute value function on a field k , and suppose that $A \subseteq V$ is balanced, absorbing, and ∞ -convex, as in the previous section. We would like to check that N_A is a semi-ultranorm on V under these conditions. We already know that N_A satisfies the homogeneity condition (6.1), and so it suffices to show that N_A satisfies the ultrametric version (6.5) of the triangle inequality. Let $r > 0$ be given, and suppose that $v_1, v_2 \in V$ satisfy

$$(14.9) \quad N_A(v_1), N_A(v_2) < r.$$

This implies that there are $t_1, t_2 \in k$ such that $|t_1|, |t_2| < r$ and

$$(14.10) \quad v_1 \in t_1 A, \quad v_2 \in t_2 A,$$

by the definition (14.1) of N_A . Let t be t_1 or t_2 , in such a way that

$$(14.11) \quad |t| = \max(|t_1|, |t_2|).$$

Using (14.10), we get that $v_1, v_2 \in t A$, since A is supposed to be balanced. It follows that

$$(14.12) \quad v_1 + v_2 \in t A,$$

because A is ∞ -convex in V , by hypothesis. Thus

$$(14.13) \quad N_A(v_1 + v_2) \leq |t| = \max(|t_1|, |t_2|) < r,$$

which implies that N_A satisfies the ultrametric version of the triangle inequality, as desired.

15 Convexity and subadditivity

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and suppose that $|\cdot|$ is not discrete on k . As in Section 12, this implies that the set (12.4) of positive values of $|\cdot|$ on k is dense in \mathbf{R}_+ with respect to the standard topology on \mathbf{R} . Also let V be a vector space over k , and let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (6.1) with respect to $|\cdot|$ on k . Remember that the corresponding open and closed balls $B_N(0, r)$ and $\overline{B}_N(0, r)$ centered at 0 in V are defined in (10.7) and (10.8), and are balanced subsets of V with respect to $|\cdot|$ on k . If either $B_N(0, r)$ or $\overline{B}_N(0, r)$ is q -convex in V with respect to $|\cdot|$ on k for any $r > 0$, then one can check that $B_N(0, r)$ and $\overline{B}_N(0, r)$ are both q -convex for every $r > 0$. This uses (10.9) and (10.10) to first go from a single positive radius to a dense set of positive radii in this situation. One can then use the remarks about unions and intersections of collections of q -convex sets at the end of Section 13 to get all positive radii, and to switch between open and closed balls. If every positive real number occurs as a value of $|\cdot|$ on k , then one can go directly from a single positive radius to all positive radii in the first step.

Suppose that for every $v_1, v_2 \in V$ with

$$(15.1) \quad N(v_1), N(v_2) < 1$$

and $t_1, t_2 \in k$ with

$$(15.2) \quad |t_1|^q + |t_2|^q \leq 1$$

we have that

$$(15.3) \quad N(t_1 v_1 + t_2 v_2) \leq 1.$$

In particular, this condition holds when either $B_N(0, 1)$ or $\overline{B}_N(0, 1)$ is q -convex in V with respect to $|\cdot|$ on k . Let $w_1, w_2 \in V$ be given, and let us check that

$$(15.4) \quad N(w_1 + w_2)^q \leq N(w_1)^q + N(w_2)^q,$$

so that N is a q -seminorm on V . If $\tau_1, \tau_2 \in k$ satisfy

$$(15.5) \quad N(w_1) < |\tau_1|, \quad N(w_2) < |\tau_2|,$$

then

$$(15.6) \quad v_1 = \tau_1^{-1} w_1, \quad v_2 = \tau_2^{-1} w_2$$

satisfy (15.1). If $\tau_3 \in k$ satisfies

$$(15.7) \quad |\tau_1|^q + |\tau_2|^q \leq |\tau_3|^q,$$

then

$$(15.8) \quad t_1 = \tau_1 / \tau_3, \quad t_2 = \tau_2 / \tau_3$$

satisfy (15.2). Thus (15.3) holds under these conditions, by hypothesis. We also have that

$$(15.9) \quad t_1 v_1 + t_2 v_2 = \tau_3^{-1} (w_1 + w_2),$$

by construction, and so (15.3) implies that

$$(15.10) \quad |\tau_3|^{-1} N(w_1 + w_2) = N(\tau_3^{-1} (w_1 + w_2)) \leq 1.$$

Equivalently,

$$(15.11) \quad N(w_1 + w_2)^q \leq |\tau_3|^q.$$

If the positive values of $|\cdot|$ on k are dense in \mathbf{R}_+ , then we can choose $\tau_1, \tau_2 \in k$ so that $|\tau_1|$ and $|\tau_2|$ are as close as we want to $N(w_1)$, $N(w_2)$, respectively. Similarly, we can choose $\tau_3 \in k$ so that $|\tau_3|^q$ is as close as we want to $|\tau_1|^q + |\tau_2|^q$, which is as close as we want to $N(w_1)^q + N(w_2)^q$. Thus (15.11) implies (15.4) in this situation, as desired.

Let A be a balanced absorbing subset of V with respect to $|\cdot|$ on k , and let N_A be the corresponding Minkowski functional on V , as in (14.1). Thus N_A satisfies the usual homogeneity condition (6.1), and the open and closed unit balls in V with respect to N_A are related to A as in (14.5). If A is also q -convex in V with respect to $|\cdot|$ on k , then (14.5) implies that N_A satisfies the condition mentioned at the beginning of the preceding paragraph. More precisely, this

means that (15.1) and (15.2) imply (15.3) when $N = N_A$. It follows that N_A is a q -seminorm on V in this situation.

Suppose now that $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, in which case the previous arguments are quite classical and can be simplified a bit. As a variant of the earlier discussion, let N be a nonnegative real-valued function on V which is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, as in (10.11). As before, the open and closed balls $B_N(0, r)$ and $\overline{B}_N(0, r)$ centered at 0 in V associated to N can be defined as in (10.7) and (10.8), and are starlike about 0 in V . If either $B_N(0, r)$ or $\overline{B}_N(0, r)$ is real q -convex for some $q \in (0, 1]$ and $r > 0$, then one can check that $B_N(0, r)$ and $\overline{B}_N(0, r)$ are both real q -convex for every $r > 0$, for the essentially same reasons as before.

Let $0 < q \leq 1$ be given, and suppose that (15.3) holds for every $v_1, v_2 \in V$ that satisfy (15.1) and nonnegative real numbers t_1, t_2 such that

$$(15.12) \quad t_1^q + t_2^q \leq 1.$$

Under these conditions, one can check that (15.4) holds for every $w_1, w_2 \in V$, so that N is q -subadditive on V . The argument is essentially the same as before, except that one can take τ_1, τ_2 , and τ_3 to be positive real numbers. One can also choose τ_3 so that equality holds in (15.7), which means that it suffices to consider $t_1, t_2 \geq 0$ such that

$$(15.13) \quad t_1^q + t_2^q = 1,$$

instead of (15.12). Note that this condition on N holds when either $B_N(0, 1)$ or $\overline{B}_N(0, 1)$ is real q -convex in V , as before.

Let $A \subseteq V$ be starlike about 0 and absorbing in V as a real vector space, and let \tilde{N}_A be the associated Minkowski functional on V , as in (14.3). Remember that \tilde{N}_A is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, and that the open and closed unit balls in V with respect to \tilde{N}_A are related to A as in (14.6). If A is also real q -convex for some $0 < q \leq 1$, then it is easy to see that \tilde{N}_A satisfies the condition described at the beginning of the previous paragraph, because of (14.6). This implies that \tilde{N}_A is q -subadditive on V .

Of course, statements like these are often formulated in terms of ordinary convexity when $q = 1$. As in Section 13, convex subsets of V that contain 0 are starlike about 0, and real 1-convex. If $A \subseteq V$ is convex and absorbing, then $0 \in A$, and \tilde{N}_A is 1-subadditive on V , as in the preceding paragraph. Similarly, if N is a nonnegative real-valued function on V which is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, then real 1-convexity of open or closed balls in V centered at 0 with respect to N is equivalent to ordinary convexity.

16 Minkowski functionals, continued

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and V be a vector space over k . Also let B, C be balanced absorbing subsets of V , and

let N_B , N_C be the corresponding Minkowski functionals on V , respectively, as in (14.1). If

$$(16.1) \quad B \subseteq C,$$

then it is easy to see that

$$(16.2) \quad N_C(v) \leq N_B(v)$$

for every $v \in V$. Similarly, suppose that $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, and that $B, C \subseteq V$ are starlike about 0, absorbing in V as a real vector space, and satisfy (16.1). If \tilde{N}_B , \tilde{N}_C are the corresponding Minkowski functionals on V , as in (14.3), then

$$(16.3) \quad \tilde{N}_C(v) \leq \tilde{N}_B(v)$$

for every $v \in V$.

Let $|\cdot|$ be a nontrivial q -absolute value function on a field k , and let N be a nonnegative real-valued function on V that satisfies the usual homogeneity condition (6.1) with respect to $|\cdot|$ on k . The nontriviality of $|\cdot|$ on k implies that the open and closed balls (10.7) and (10.8) in V centered at 0 with radius $r > 0$ with respect to N are absorbing in V , and we have also seen that they are balanced in V . Let us take

$$(16.4) \quad C = \overline{B}_N(0, 1)$$

for the moment, and consider the corresponding Minkowski functional N_C on V , as in (14.1). Suppose that $v \in V$ and $t \in k$ satisfy $v \in tC$. If $t = 0$, then it follows that $v = 0$. Otherwise, if $t \neq 0$, then

$$(16.5) \quad v \in tC = \overline{B}_N(0, |t|),$$

using (10.10) in the second step. This implies that

$$(16.6) \quad N(v) \leq |t|,$$

which also works when $t = 0$. Taking the infimum over t , we get that

$$(16.7) \quad N(v) \leq N_C(v)$$

for every $v \in V$ under these conditions.

Suppose now that $|\cdot|$ is not discrete on k , so that the set (12.4) of positive values of $|\cdot|$ on k is dense in \mathbf{R}_+ with respect to the standard topology on \mathbf{R} , as in Section 12. As before,

$$(16.8) \quad B = B_N(0, 1)$$

is balanced and absorbing in V , and we let N_B be the corresponding Minkowski functional on V . Let $v \in V$ be given, and suppose that $t \in k$ satisfies $N(v) < |t|$, so that

$$(16.9) \quad v \in B_N(0, |t|) = |t|B,$$

using (10.9) in the second step. This implies that

$$(16.10) \quad N_B(v) \leq |t|,$$

and hence that

$$(16.11) \quad N_B(v) \leq N(v),$$

by taking the infimum over t in (16.10). If C is as in (16.4), then (16.1) holds, and we get that

$$(16.12) \quad N(v) = N_B(v) = N_C(v)$$

for every $v \in V$, by combining (16.2), (16.7), and (16.11).

Similarly, let $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function again, and let N be a nonnegative real-valued function on V that is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, as in (10.11). Also let B, C be as in (16.4) and (16.8), so that B, C are starlike about 0, absorbing in V as a real vector space, and satisfy (16.1). If \tilde{N}_B, \tilde{N}_C are the corresponding Minkowski functionals on V , then one can check that

$$(16.13) \quad N(v) = \tilde{N}_B(v) = \tilde{N}_C(v)$$

for every $v \in V$, as in the previous two paragraphs. More precisely, in this situation, one should restrict one's attention to nonnegative real numbers t in the earlier arguments.

17 Continuity of semimetrics

Let X be a set, and let $d(x, y)$ be a q -semimetric on X for some $q > 0$. Thus

$$(17.1) \quad d(x, z)^q - d(y, z)^q \leq d(x, y)^q$$

for every $x, y, z \in X$, by the q -semimetric version (7.1) of the triangle inequality. Similarly,

$$(17.2) \quad d(y, z)^q - d(x, z)^q \leq d(x, y)^q$$

for every $x, y, z \in X$, and hence

$$(17.3) \quad |d(x, z)^q - d(y, z)^q| \leq d(x, y)^q,$$

using the standard absolute value function on \mathbf{R} on the left side of (17.3). This implies that $d(x, z)$ is continuous as a real-valued function of $x \in X$ for each $z \in X$, with respect to the topology determined on X by $d(\cdot, \cdot)$ as in Sections 1 and 7, and using the standard topology on \mathbf{R} in the range of this function. Of course, this was also implicit in some of the earlier discussions. Using the analogous estimate in both variables, we get that

$$\begin{aligned} (17.4) \quad |d(x, w)^q - d(y, z)^q| &\leq |d(x, w)^q - d(x, z)^q| + |d(x, z)^q - d(y, z)^q| \\ &\leq d(w, z)^q + d(x, y)^q \end{aligned}$$

for every $w, x, y, z \in X$. In particular, this implies that $d(x, w)$ is continuous on $X \times X$, with respect to the product topology associated to the topology determined on X by $d(\cdot, \cdot)$.

Let τ be a topology on X . Suppose that for each $u \in X$ and $\epsilon > 0$ there is an open set $U \subseteq X$ with respect to τ such that $u \in U$ and

$$(17.5) \quad d(u, v) < \epsilon$$

for every $v \in U$. Equivalently, (17.5) says that

$$(17.6) \quad U \subseteq B_d(u, \epsilon),$$

where $B_d(u, \epsilon)$ is the open ball in X centered at u with radius ϵ with respect to $d(\cdot, \cdot)$, as in (1.5). This is also the same as saying that u is an element of the interior of $B_d(u, \epsilon)$ with respect to τ for every $u \in X$ and $\epsilon > 0$. This condition implies that every open set in X with respect to the topology on X determined by $d(\cdot, \cdot)$ is an open set with respect to τ as well. Conversely, suppose that every open set in X with respect to the topology determined by $d(\cdot, \cdot)$ is an open set in X with respect to τ too. Remember that open balls in X are open sets in X with respect to the topology determined by $d(\cdot, \cdot)$, as in Sections 1 and 7. In this case, open balls in X with respect to $d(\cdot, \cdot)$ are also open sets with respect to τ , which obviously implies the previous condition.

The condition described at the beginning of the preceding paragraph is the same as saying that for each $u \in X$, $d(u, v)$ is continuous as a real-valued function of $v \in X$ with respect to τ at u . This implies that $d(x, y)$ is continuous as a real-valued function of x or y with respect to τ on X , and in fact that $d(x, y)$ is continuous with respect to the product topology on $X \times X$ associated to τ on each copy of X . This can be derived from (17.3) and (17.4) as before. Alternatively, this can be obtained from the analogous continuity properties with respect to the topology determined on X by $d(\cdot, \cdot)$, and the fact that every open set in X with respect to the topology determined by $d(\cdot, \cdot)$ is also an open set with respect to τ .

Now let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . If N is a q -seminorm on V with respect to $|\cdot|$ on k , then we have that

$$(17.7) \quad |N(v)^q - N(w)^q| \leq N(v - w)^q$$

for every $v, w \in V$. This can be verified in the same way as for (17.3). This can also be considered as a special case of (17.3), using the q -semimetric associated to N on V , as in (9.2). As before, this implies in particular that $N(v)$ is continuous as a real-valued function on V , with respect to the topology determined on V by the q -semimetric associated to N as in (9.2).

18 Commutative topological groups

Let A be a commutative group, with the group operations expressed additively. Suppose that A is also equipped with a topology. We say that A is a *topological*

group if the group operations on A are continuous. More precisely, this means that addition on A should be continuous as a mapping from $A \times A$ into A , where $A \times A$ is equipped with the product topology associated to the given topology on A . Similarly,

$$(18.1) \quad x \mapsto -x$$

should be continuous as a mapping from A into itself, which implies that this mapping is a homeomorphism, since it is its own inverse.

Continuity of addition on A implies that

$$(18.2) \quad x \mapsto x + a$$

is a continuous mapping from A into itself for each $a \in A$. This corresponds to continuity of addition on A in each variable separately, instead of joint continuity of addition as a mapping from $A \times A$ into A . Of course, the inverse of the translation mapping (18.2) is given by translation by $-a$, so that continuity of translations implies that the translation mappings (18.2) are homeomorphisms on A . If A is equipped with a topology for which the translation mappings (18.2) are continuous, and if addition on A is continuous as a mapping from $A \times A$ into A with respect to the product topology on $A \times A$ at the point $(0,0)$ in $A \times A$, then one can check that addition on A is continuous as a mapping from $A \times A$ into A at every point in $A \times A$. Similarly, if the translation mappings (18.2) are continuous on A , and if (18.1) is continuous at 0, then (18.1) is continuous everywhere on A .

Let us suppose for the rest of the section that A is a commutative topological group. If $a \in A$ and $B \subseteq A$, then put

$$(18.3) \quad a + B = B + a = \{a + b : b \in B\},$$

which is the same as the image of B under the translation mapping (18.2). If $B, C \subseteq A$, then we put

$$(18.4) \quad B + C = \{b + c : b \in B, c \in C\} = \bigcup_{b \in B} (b + C) = \bigcup_{c \in C} (B + c).$$

In particular, if either B or C is an open set in A , then $B + C$ is an open set in A as well, because it is a union of open sets in A . Let us also put

$$(18.5) \quad -C = \{-c : c \in C\},$$

$$(18.6) \quad b - C = b + (-C),$$

and

$$(18.7) \quad B - C = B + (-C)$$

for every $b \in A$ and $B, C \subseteq A$.

Let E be any subset of A , and let $V \subseteq A$ be an open set that contains 0. We would like to check that

$$(18.8) \quad \overline{E} \subseteq E + V,$$

where \overline{E} is the closure of E in V , as usual. If $x \in \overline{E}$, then

$$(18.9) \quad (x - V) \cap E \neq \emptyset,$$

because $x - V$ is an open set in V that contains x . This is the same as saying that $x \in E + V$, which implies (18.8). Similarly, if $x \in E + V$ for every open set $V \subseteq A$ that contains 0, then $x \in \overline{E}$, which implies that

$$(18.10) \quad \overline{E} = \bigcap\{E + V : V \subseteq A \text{ is an open set with } 0 \in V\}.$$

Let W be an open subset of A that contains 0. Continuity of addition on A as a mapping from $A \times A$ into A at $(0, 0)$ says exactly that there are open sets $U, V \subseteq A$ that contain 0 and satisfy

$$(18.11) \quad U + V \subseteq W.$$

In particular, this implies that

$$(18.12) \quad \overline{U} \subseteq W,$$

as in (18.8). If x is any element of A , then it follows that every open set in A that contains x also contains the closure of another open subset of A that contains x , by continuity of translations on A . This means that A is regular as a topological space in the strict sense, without including the first or 0th separation condition.

Sometimes the condition that $\{0\}$ be a closed set in A is included in the definition of a commutative topological group. This implies that every subset of A with exactly one element is a closed set, by continuity of translations, so that A satisfies the first separation condition. Combining this with the remarks in the preceding paragraph, we get that A is regular in the stronger sense that includes the first separation condition, which is sometimes referred to as the third separation condition. In particular, this implies that A is Hausdorff as a topological space.

Here is another way to look at the Hausdorff property of A when $\{0\}$ is a closed set in A . Let $x, y \in A$ be given, with $x \neq y$, and put

$$(18.13) \quad W = A \setminus \{x - y\},$$

so that $0 \in W$. Note that W is an open set in A , since A satisfies the first separation condition. Thus there are open sets $U, V \subseteq A$ that contain 0 and satisfy (18.11), which means that

$$(18.14) \quad x - y \notin U + V.$$

Equivalently, this implies that

$$(18.15) \quad (x - U) \cap (y + V) = \emptyset,$$

and hence that A is Hausdorff as a topological space, because $x - U$ and $y + V$ are disjoint open subsets of A that contain x and y , respectively.

Similarly, let $x \in A$ and $E \subseteq A$ be given, with $x \notin E$, and put

$$(18.16) \quad W = A \setminus (x - E),$$

so that $0 \in W$. If E is a closed subset of A , then $x - E$ is a closed set in A too, and hence W is an open set in A . This implies that there are open sets $U, V \subseteq A$ that contain 0 and satisfy (18.11), as before. In this case, (18.11) says that

$$(18.17) \quad (U + V) \cap (x - E) = \emptyset,$$

which implies that

$$(18.18) \quad (x - U) \cap (E + V) = \emptyset.$$

This is another way to look at the regularity of A as a topological space, since $x - U$ and $E + V$ are disjoint open subsets of A that contain x and E , respectively.

19 Translation-invariant semimetrics

Let A be a commutative group, and let $d(x, y)$ be a semimetric on A . If

$$(19.1) \quad d(x + a, y + a) = d(x, y)$$

for every $a, x, y \in A$, then we say that $d(\cdot, \cdot)$ is *invariant under translations* on A . In this case, the translation mappings (18.2) automatically define homeomorphisms on A with respect to the topology on A determined by $d(\cdot, \cdot)$. One can also check that addition on A is continuous as a mapping from $A \times A$ into A under these conditions, using the triangle inequality. Similarly, it is easy to see that (18.1) is continuous on A , because $d(x, y)$ is symmetric in x and y . This means that A satisfies the requirements of a commutative topological group with respect to the topology determined by $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a translation-invariant metric on A , then A is also Hausdorff with respect to this topology. Of course, invariance under translations can be defined for q -semimetrics on A in the same way, for any $q > 0$. Equivalently, a q -semimetric $d(x, y)$ on A is invariant under translations if and only if $d(x, y)^q$ is invariant under translations as a semimetric on A , in which case there are analogues of the previous statements for the corresponding topology.

Let \mathcal{M} be a collection of q -semimetrics on A , where $q > 0$ is allowed to depend on the element of \mathcal{M} . This leads to a topology on A , as in Sections 2 and 7. If the elements of \mathcal{M} are invariant under translations on A , then the group operations on A are continuous with respect to this topology, as in the preceding paragraph, so that A becomes a commutative topological group. If \mathcal{M} is also nondegenerate on A , then we have seen that A is Hausdorff with respect to this topology. If A is any commutative topological group, then it is well known that the topology on A corresponds to a collection of translation-invariant semimetrics on A in this way. Note that any commutative group is a topological group with respect to the discrete topology. The discrete metric on any group is invariant under translations, and determines the discrete topology on the group.

Let A be a commutative topological group, and let $d(x, y)$ be a translation-invariant q -semimetric on A for some $q > 0$. Suppose that $d(0, v)$ is continuous as a real-valued function of $v \in A$ at 0, with respect to the standard topology on \mathbf{R} in the range of this function. This implies that for each $u \in A$, $d(u, v)$ is continuous as a real-valued function of $v \in A$ at u , because of translation-invariance of $d(\cdot, \cdot)$ and the topology on A . As in Section 17, it follows that $d(x, y)$ is continuous as a real-valued function of x or y on A , and in fact that $d(x, y)$ is continuous as a real-valued function on $A \times A$ with respect to the corresponding product topology. This condition also implies that every open set in A with respect to the topology determined by $d(\cdot, \cdot)$ is an open set in A with respect to its given topology, as before.

If the topology on A is determined by a collection \mathcal{M} of translation-invariant q -semimetrics on A , then each element of \mathcal{M} automatically has the property described in the preceding paragraph. If $d(x, y)$ is any other translation-invariant q -semimetric on A with this property, then one can add $d(x, y)$ to \mathcal{M} , and get the same topology on A . If A is any commutative topological group, then one might like to find a collection \mathcal{M} of translation-invariant q -semimetrics on A that determines the same topology on A , as mentioned earlier. Each element of \mathcal{M} should be compatible with the given topology on A , as in the previous paragraph. The point of the theorem mentioned earlier is that one can always find enough translation-invariant semimetrics on A that are compatible with the given topology on A to generate this topology on A .

20 Topological vector spaces

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . Remember that $|x - y|$ defines a q -metric on k , which determines a topology on k in the usual way. We say that V is a *topological vector space* with respect to $|\cdot|$ on k if V is equipped with a topology for which the vector space operations are continuous. More precisely, the continuity of addition on V means that addition defines a continuous mapping from $V \times V$ into V , with respect to the product topology on $V \times V$ associated to the given topology on V . Similarly, continuity of scalar multiplication on V means that scalar multiplication defines a continuous mapping from $k \times V$ into V , where $k \times V$ is equipped with the product topology associated to the topology on k mentioned earlier and the given topology on V .

In particular, continuity of scalar multiplication implies that for each $t \in k$,

$$(20.1) \quad v \mapsto t v$$

is a continuous mapping on V . This includes the continuity of

$$(20.2) \quad v \mapsto -v$$

on V , by taking $t = -1$ in k in (20.1). Of course, vector spaces are commutative groups with respect to addition, and this implies that topological vector spaces

are topological groups with respect to addition. If $\{0\}$ is a closed set V , then it follows that V is Hausdorff as a topological space, as in Section 18. This additional condition is sometimes included in the definition of a topological vector space, as before.

If $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, then this corresponds to the usual notion of a real or complex topological vector space. If $|\cdot|$ is a nontrivial absolute value function on any field k , then this definition corresponds to the one in [21]. The restriction to $q = 1$ is not too serious, since $|x|^q$ defines an absolute value function on k when $|x|$ is a q -absolute value function on k , and which leads to the same topology on k . If $|\cdot|$ is the trivial absolute value function on k , then the corresponding topology on k is discrete, and the continuity of scalar multiplication on V as a mapping from $k \times V$ into V with respect to the product topology on $k \times V$ is the same as the continuity of (20.1) on V for each $t \in k$. One may wish to exclude this case in some situations, as in [21].

Let $|\cdot|$ be a q -absolute value function on a field k for some $q > 0$ again, and let V be a topological vector space over k . Note that (20.1) is a homeomorphism on V for each $t \in k$ with $t \neq 0$, since the inverse mapping is given by multiplication by $1/t$. Let W be an open subset of V that contains 0. Continuity of scalar multiplication at $(0, 0)$ in $k \times V$ implies that there is an open set $U \subseteq V$ with $0 \in U$ and a $\delta > 0$ such that

$$(20.3) \quad tU \subseteq W$$

for every $t \in k$ with $|t| < \delta$, where tU is as in (10.1). This statement is vacuous when $|\cdot|$ is the trivial absolute value function on k , and so let us suppose for the moment that $|\cdot|$ is nontrivial on k . This implies that there are $t \in k$ with $t \neq 0$ and $|t| < \delta$, and we put

$$(20.4) \quad U_1 = \bigcup\{tU : t \in k, 0 < |t| < \delta\},$$

which contains 0 because $0 \in U$. We also have that

$$(20.5) \quad U_1 \subseteq W,$$

by (20.3), and that U_1 is an open set in V , since it is a union of open sets. By construction, U_1 is a balanced subset of V too, as in Section 10.

Let us rephrase this a bit, as follows. If W is an open set in V that contains 0, and if $|\cdot|$ is nontrivial on k , then there is an open set $\tilde{U} \subseteq V$ such that $0 \in \tilde{U}$ and

$$(20.6) \quad t\tilde{U} \subseteq W$$

for every $t \in k$ with $|t| \leq 1$. More precisely, if U satisfies the conditions described in the preceding paragraph, then one can take \tilde{U} to be U_1 , as in (20.4). Conversely, if \tilde{U} satisfies these conditions, then one can simply take U to be \tilde{U} in the previous paragraph. In this case,

$$(20.7) \quad \tilde{U}_1 = \bigcup\{t\tilde{U} : t \in k, |t| \leq 1\}$$

is a nonempty balanced open subset of V that is contained in W , as before.

If $|\cdot|$ is the trivial absolute value function on k , then the existence of \tilde{U} as in (20.6) still makes sense, even if it is not necessarily included in the definition of a topological vector space. It would also be stronger than before, in the sense that it applies to every $t \in k$, and similarly balanced subsets of V would be invariant under multiplication by any $t \in k$ with $t \neq 0$, as in Section 10. If $|\cdot|$ is any q -absolute value function on k and V is a topological vector space with respect to $|\cdot|$ on k , then V is also a topological vector space with respect to the trivial absolute value function on k , because (20.1) is still continuous on V for every $t \in k$. In this case, (20.6) would not normally work with respect to the trivial absolute value function on k . In particular, one can take $V = k$ with the topology associated to a nontrivial absolute value function on k .

Let $|\cdot|$ be any q -absolute value function on a field k again. If V is a topological vector space over k , then continuity of scalar multiplication implies that

$$(20.8) \quad t \mapsto t v$$

is a continuous mapping from k into V for every $v \in V$. Let W be an open subset of V that contains 0, and let $v \in V$ be given. The continuity of (20.8) at 0 implies that there is a $\delta(v) > 0$ such that

$$(20.9) \quad t v \in W$$

for every $t \in k$ with $|t| < \delta(v)$. This is the same as saying that W is absorbing in V when $|\cdot|$ is nontrivial on k , as in Section 11. If $|\cdot|$ is the trivial absolute value function on k , then this condition on W is vacuous, and we have seen that V is the only absorbing subset of itself in this case.

21 Compatible seminorms

Let k be a field equipped with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . Also let \mathcal{N} be a collection of q -seminorms on V with respect to $|\cdot|$ on k . More precisely, one can let $q > 0$ depend on the element of \mathcal{N} , as long as $|\cdot|$ is a q -absolute value function on k for each such q . Every element of \mathcal{N} determines a q -semimetric on V in the usual way, as in (9.2). Note that these q -semimetrics are automatically invariant under translations on V . Consider the topology on V determined by the collection \mathcal{M} of q -semimetrics associated to the elements of \mathcal{N} . One can check that scalar multiplication on V defines a continuous mapping from $k \times V$ into V with respect to this topology on V , using the product topology on $k \times V$ corresponding to this topology on V and the topology on k determined by the q -metric associated to $|\cdot|$. This implies that V is a topological vector space with respect to $|\cdot|$ on k under these conditions. If \mathcal{N} is nondegenerate on V , then V is Hausdorff with respect to this topology, as before. If $|\cdot|$ is the trivial absolute value function on k , then this topology on V also satisfies the condition (20.6) discussed in the previous section, because of the corresponding homogeneity property of q -seminorms on V .

Suppose now that V is equipped with a topology that makes it a topological vector space with respect to $|\cdot|$ on k , and let N be a q -seminorm with respect to $|\cdot|$ on k . Suppose that N is continuous at 0 as a real-valued function on V , where \mathbf{R} is equipped with the standard topology in the range of this function. It is easy to see that this implies that N is continuous on all of V , using (17.7). In particular, it follows that open balls in V with respect to N are open sets in V with respect to the given topology on V . One can look at this in the context of Section 19 as well, since V is a commutative topological group with respect to addition. Let $d(v, w)$ be the q -semimetric on V associated to N as in (9.2), which is automatically invariant under translations on V , as mentioned earlier. The hypothesis that N be continuous at 0 is the same as saying that $d(0, w)$ is continuous as a real-valued function of $w \in V$ at 0. As in Section 19, this implies that $d(v, w)$ is continuous in v and w , which corresponds to the continuity of N on V . This also implies that open subsets of V with respect to the topology determined by $d(\cdot, \cdot)$ are open sets with respect to the given topology on V , as before.

As in Section 17, the condition that N be continuous at 0 is the same as saying that for each $\epsilon > 0$, 0 is an element of the interior of $B_N(0, \epsilon)$ in V , where $B_N(0, \epsilon)$ is as in (10.7). This implies that N is continuous on all of V , as in the preceding paragraph, and hence that $B_N(0, r)$ is an open set in V for every $r > 0$. Remember that

$$(21.1) \quad t B_N(0, \epsilon) = B_N(0, |t| \epsilon),$$

for every $\epsilon > 0$ and $t \in k$ with $t \neq 0$, as in (10.9). If 0 is an element of the interior of $B_N(0, \epsilon)$ in V for some $\epsilon > 0$, then it follows that 0 is also an element of the interior of (21.1) for every $t \in k$ with $t \neq 0$. This implies that N is continuous at 0 when $|\cdot|$ is not the trivial absolute value function on k .

Let A be a balanced absorbing subset of V , and let N_A be the corresponding Minkowski functional, as in (14.1). Thus

$$(21.2) \quad N_A(v) \leq |t|$$

for every $t \in k$ and $v \in tA$, by definition of $N_A(v)$. If $|\cdot|$ is nontrivial on k , and if 0 is an element of the interior of A in V , then it follows from (21.2) that N_A is continuous at 0 as a real-valued function on V . Of course, if $|\cdot|$ is trivial on k , then V is the only absorbing subset of itself, and N_V is equal to 0 on all of V , as in Section 14. Conversely, if N_A is continuous at 0 on V , then 0 is an element of the interior of $B_{N_A}(0, 1)$ in V , where $B_{N_A}(0, 1)$ is defined as in (10.7). We have also seen that $B_{N_A}(0, 1)$ is contained in A when A is balanced, as in (14.5). This implies that 0 is an element of the interior of A in V when N_A is continuous at 0 on V .

22 Subadditive functions

Let us take $k = \mathbf{R}$ with the standard absolute value function in this section, and let V be a vector space over \mathbf{R} . Also let N be a nonnegative real-valued

function on V which is homogeneous of degree 1 with respect to multiplication by nonnegative real numbers, as in (10.11). Suppose that N is q -subadditive on V for some $q > 0$, as in (13.5). More precisely, we may as well take $0 < q \leq 1$ here, so that the standard absolute value function on \mathbf{R} is a q -absolute value function, as in Section 8. Observe that $N(-v)$ satisfies these same properties on V , which is to say that $N(-v)$ is also homogeneous with respect to multiplication by nonnegative real numbers and q -subadditive. This implies that

$$(22.1) \quad N^{sym}(v) = \max(N(v), N(-v))$$

has the same properties too, because the maximum of two q -subadditive functions is q -subadditive. By construction,

$$(22.2) \quad N^{sym}(-v) = N^{sym}(v)$$

for every $v \in V$, so that \tilde{N} is a q -seminorm on V . As in Section 17, we have that

$$(22.3) \quad N(v)^q - N(w)^q \leq N(v - w)^q$$

and

$$(22.4) \quad N(w)^q - N(v)^q \leq N(w - v)^q$$

for every $v, w \in V$, by q -subadditivity. It follows that

$$(22.5) \quad |N(v)^q - N(w)^q| \leq N^{sym}(v - w)^q$$

for every $v, w \in V$, so that N is continuous as a real-valued function on V with respect to the topology determined on V by the q -semimetric associated to N^{sym} .

Suppose now that V is equipped with a topology which makes it into a topological vector space over \mathbf{R} . If $N(v)$ is continuous at 0 as a real-valued function on V with respect to this topology, then $N(-v)$ is also continuous at 0, and hence $N^{sym}(v)$ is continuous at 0. This implies that N is continuous on all of V , by (22.5). In particular, this means that open balls in V with respect to N are open sets in V . The continuity of N^{sym} at 0 on V also implies that N^{sym} is continuous on all of V , as in the previous section.

Let A be a subset of V that is starlike about 0 and absorbing, and let \tilde{N}_A be the corresponding Minkowski functional, as in (14.3). It is easy to see that $-A$ is starlike about 0 and absorbing in V too, and we let \tilde{N}_{-A} be the Minkowski functional associated to $-A$ on V as in (14.3). Similarly,

$$(22.6) \quad A^{sym} = A \cap (-A)$$

is starlike about 0 and absorbing in V , and in fact A^{sym} is balanced in V , since it is automatically symmetric about 0. Thus the Minkowski functional $N_{A^{sym}}$ associated to A^{sym} as in (14.1) is the same as its analogue $\tilde{N}_{A^{sym}}$ as in (14.3). One can check that

$$(22.7) \quad \tilde{N}_A(-v) = \tilde{N}_{-A}(v)$$

and

$$(22.8) \quad N_{A^{sym}}(v) = \max(\tilde{N}_A(v), \tilde{N}_A(-v))$$

for every $v \in V$.

As in the previous section,

$$(22.9) \quad \tilde{N}_A(v) \leq t$$

for every $v \in tA$ when $t \geq 0$, by the definition of $\tilde{N}_A(v)$. If 0 is an element of the interior of A in V , then it follows that $\tilde{N}_A(v)$ is continuous at 0 on V , as before. Conversely, if $\tilde{N}_A(v)$ is continuous at 0 on V , then 0 is an element of the interior of $B_{\tilde{N}_A}(0, 1)$ in V , where $B_{\tilde{N}_A}(0, 1)$ is as defined in (10.7). This implies that 0 is an element of the interior of A in V , because $B_{\tilde{N}_A}(0, 1)$ is contained in A when A is starlike about 0, as in (14.6). Of course, if 0 is an element of the interior of A in V , then 0 is also an element of the interior of $-A$, which implies that 0 is an element of the interior of A^{sym} as well.

23 Cartesian products

Let I be a nonempty set, let X_j be a set for each $j \in I$, and let

$$(23.1) \quad X = \prod_{j \in I} X_j$$

be the Cartesian product of the X_j 's. If $x \in X$ and $j \in I$, then it will be convenient to let x_j be the j th coordinate of x in X_j . Let \mathcal{M}_j be a collection of q -semimetrics on X_j for each $j \in I$, where $q > 0$ is allowed to depend on the element of \mathcal{M}_j , as before. If $d_j(\cdot, \cdot)$ is an element of \mathcal{M}_j for some $j \in I$, then it is easy to see that

$$(23.2) \quad \hat{d}_j(x, y) = d_j(x_j, y_j)$$

defines a q -semimetric on X , with the same q as for $d_j(\cdot, \cdot)$. Put

$$(23.3) \quad \widehat{\mathcal{M}}_j = \{\hat{d}_j(\cdot, \cdot) : d_j(\cdot, \cdot) \in \mathcal{M}_j\}$$

for each $j \in I$, and

$$(23.4) \quad \mathcal{M} = \bigcup_{j \in I} \widehat{\mathcal{M}}_j.$$

Let X_j be equipped with the topology associated to \mathcal{M}_j as in Sections 2 and 7, for each $j \in I$. One can check that the topology on X associated to \mathcal{M} in this way is the same as the corresponding product topology on X . Note that \mathcal{M} is nondegenerate on X when \mathcal{M}_j is nondegenerate on X_j for every $j \in I$.

Now let A_j be a commutative group for each $j \in I$, and let

$$(23.5) \quad A = \prod_{j \in I} A_j$$

be the Cartesian product of the A_j 's. This is also a commutative group, where the group operations are defined coordinatewise. This group is known as the *direct product* of the A_j 's. If A_j is a commutative topological group for each $j \in I$, then A is a commutative topological group too, with respect to the corresponding product topology. Let \mathcal{M}_j be a collection of translation-invariant q -semimetrics on A , and let $\widehat{\mathcal{M}}_j$ be the corresponding collection of q -semimetrics on A for each $j \in I$, as in (23.3). Note that the elements of $\widehat{\mathcal{M}}_j$ are invariant under translations on A for each j , so that the union \mathcal{M} of the $\widehat{\mathcal{M}}_j$'s consists of translation-invariant q -semimetrics on A as well. If A_j is equipped with the topology determined by \mathcal{M}_j for each $j \in I$, then the topology on A associated to \mathcal{M} is the same as the corresponding product topology, as in the previous paragraph.

Let k be a field, and let V_j be a vector space over k for each $j \in I$. The Cartesian product

$$(23.6) \quad V = \prod_{j \in I} V_j$$

is a vector space over k too, with respect to coordinatewise addition and scalar multiplication. This vector space is known as the *direct product* of the V_j 's. Let $|\cdot|$ be a q -absolute value function on k for some $q > 0$, and suppose that V_j is a topological vector space with respect to $|\cdot|$ on k for each $j \in I$. Under these conditions, one can check that V is a topological vector space too, with respect to the corresponding product topology.

Let \mathcal{N}_j be a collection of q -seminorms on V_j for each $j \in I$, where $q > 0$ is allowed to depend on the element of \mathcal{N}_j , as long as $|\cdot|$ is a q -absolute value function on k . If N_j is an element of \mathcal{N}_j for some $j \in I$, then

$$(23.7) \quad \widehat{N}_j(v) = N_j(v_j)$$

defines a q -seminorm on V with the same q as for N_j , where $v_j \in V_j$ is the j th coordinate of $v \in V$, as before. In analogy with (23.3) and (23.4), put

$$(23.8) \quad \widehat{\mathcal{N}}_j = \{\widehat{N}_j : N_j \in \mathcal{N}_j\}$$

for each $j \in I$, and

$$(23.9) \quad \mathcal{N} = \bigcup_{j \in I} \widehat{\mathcal{N}}_j.$$

Let \mathcal{M}_j be the collection of q -semimetrics on V_j corresponding to elements of \mathcal{N}_j as in (9.2) for each $j \in I$. If $\widehat{\mathcal{M}}_j$ is associated to \mathcal{M}_j as in (23.3) for each $j \in I$, then $\widehat{\mathcal{M}}_j$ is the same as the collection of q -semimetrics on V that correspond to elements of $\widehat{\mathcal{N}}_j$ as in (9.2) for each $j \in I$. Similarly, if \mathcal{M} is as in (23.4), then \mathcal{M} is the same as the collection of q -semimetrics on V that correspond to elements of \mathcal{N} as in (9.2) for each $j \in I$. If V_j is equipped with the topology determined by \mathcal{N}_j for each $j \in I$, then it follows that the topology on V associated to \mathcal{N} is the same as the corresponding product topology.

24 Bounded semimetrics

Let X be a set, and let $d(x, y)$ be a q -semimetric on X for some $q > 0$. If r_0 is any positive real number, then it is easy to see that

$$(24.1) \quad d'(x, y) = \min(d(x, y), r_0)$$

is also a q -semimetric on X . We also have that

$$(24.2) \quad B_{d'}(x, r) = B_d(x, r)$$

for every $x \in X$ and $0 < r \leq r_0$, and that

$$(24.3) \quad B_{d'}(x, r) = X$$

for every $x \in X$ when $r > r_0$, where the open balls are defined as in (1.5). This implies that $d(x, y)$ and $d'(x, y)$ determine the same topologies on X , since only balls of small radius are important for the topology. Similarly, if X is already equipped with a topology τ , then continuity properties of $d(x, y)$ with respect to τ as in Section 17 are equivalent to the analogous continuity properties of (24.1) with respect to τ .

Let A be a commutative group, and suppose that $d(x, y)$ is a q -semimetric on A for some $q > 0$ that is invariant under translations on A . In this case, (24.1) is also invariant under translations on A for every $r_0 > 0$. If A is a commutative topological group, then continuity properties of $d(x, y)$ on A as in Section 17 can be reformulated as continuity conditions at 0, as in Section 19. These continuity conditions are equivalent to their analogues for (24.1), as in the preceding paragraph. Note that

$$(24.4) \quad d(x, y) = d(x - (x + y), y - (x + y)) = d(-y, -x) = d(-x, -y)$$

for every $x, y \in A$, using invariance under translations in the first step.

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . Also let N be a q -seminorm on V , and let $d(v, w)$ be the associated q -semimetric, as in (9.2). Thus

$$(24.5) \quad d(tv, tw) = N(tv - tw) = |t|N(v - w) = |t|d(v, w)$$

for every $v, w \in V$ and $t \in k$, using the homogeneity of N in the second step. In particular, this implies that

$$(24.6) \quad d(tv, tw) \leq d(v, w)$$

for every $v, w \in V$ when $|t| \leq 1$, with equality when $|t| = 1$. If $d'(v, w)$ is defined as in (24.1) for some $r_0 > 0$, then it follows that

$$(24.7) \quad d'(tv, tw) \leq d'(v, w)$$

for every $v, w \in V$ when $t \in k$ satisfies $|t| \leq 1$, with equality when $|t| = 1$.

Now let $d(v, w)$ be any q -semimetric on V that is invariant under translations. Observe that $d(v, w)$ satisfies (24.6) when $|t| \leq 1$ exactly when open and closed balls in V with respect to d centered at 0 are balanced subsets of V . If $d(v, w)$ does not already have this property, then one may wish to replace $d(v, w)$ with

$$(24.8) \quad \tilde{d}(v, w) = \sup\{d(tv, tw) : t \in k, |t| \leq 1\},$$

at least when the supremum is finite. In this case, it is easy to see that (24.8) is also a q -semimetric on V that is invariant under translations. By construction,

$$(24.9) \quad \tilde{d}(av, aw) \leq \tilde{d}(v, w)$$

for every $v, w \in V$ and $a \in k$ with $|a| \leq 1$, and

$$(24.10) \quad d(v, w) \leq \tilde{d}(v, w)$$

for every $v, w \in V$. Of course, if $d(v, w)$ is bounded on V , then the supremum in (24.8) is finite, with the same upper bounds as for $d(v, w)$. Otherwise, one can first replace $d(v, w)$ with (24.1) for some $r_0 > 0$, to ensure that it is bounded.

Suppose that V is a topological vector space with respect to $|\cdot|$ on k , and that $d(v, w)$ is compatible with the topology on V , in the sense that $d(0, w)$ is continuous as a real-valued function of $w \in V$ at 0. Equivalently, this means that for each $\epsilon > 0$, 0 is an element of the interior of $B_d(0, \epsilon)$ in V , where $B_d(0, \epsilon)$ is the open ball in V centered at 0 with radius ϵ with respect to $d(v, w)$, as in (1.5). This is the same as saying that for each $\epsilon > 0$ there is an open set $W(\epsilon) \subseteq V$ such that $0 \in W(\epsilon)$ and

$$(24.11) \quad W(\epsilon) \subseteq B_d(0, \epsilon).$$

If $|\cdot|$ is nontrivial on k , then for each $\epsilon > 0$ there is an open set $U(\epsilon) \subseteq V$ such that $0 \in U(\epsilon)$ and

$$(24.12) \quad tU(\epsilon) \subseteq W(\epsilon)$$

for every $t \in k$ with $|t| \leq 1$, as in (20.6). It follows that

$$(24.13) \quad d(0, tu) < \epsilon$$

for every $\epsilon > 0$, $u \in U(\epsilon)$, and $t \in k$ with $|t| \leq 1$, by combining (24.11) and (24.12).

Let $\tilde{d}(v, w)$ be as in (24.8), and suppose that the supremum is finite for every $v, w \in V$. As before, this can always be arranged by replacing $d(v, w)$ with (24.1) for some $r_0 > 0$, which would not affect the continuity properties of $d(v, w)$. Using (24.13), we get that

$$(24.14) \quad \tilde{d}(0, u) \leq \epsilon$$

for every $\epsilon > 0$ and $u \in U(\epsilon)$. This shows that $\tilde{d}(0, u)$ is continuous as a real-valued function of $u \in V$ at 0 under these conditions. Remember that this

implies additional continuity properties of $\tilde{d}(v, w)$ on V , as in Sections 17 and 19. If $|\cdot|$ is the trivial absolute value function on k , then (20.6) may be considered as an additional hypothesis on the topology of V . With this additional hypothesis, the rest of the previous argument goes through as before.

As mentioned in Section 19, there are well known methods for constructing translation-invariant semimetrics on a commutative topological group that are compatible with the given topology. In particular, if V is a topological vector space with respect to $|\cdot|$ on k , as before, then this can be applied to V as a commutative topological group with respect to addition. Under suitable conditions, the arguments in the preceding paragraphs can be used to get semimetrics on V that also satisfy (24.9). However, translation-invariant semimetrics on V with this property are typically obtained directly from the original construction under the same conditions. The main point is to use balanced neighborhoods of 0 in V in the construction of the semimetrics, when $|\cdot|$ is nontrivial on k , or when $|\cdot|$ is trivial on k and the topology on V satisfies (20.6).

25 Metrization

Let \mathcal{M} be a nonempty collection of q -seminorms on a set X for some $q > 0$. Although we have often allowed q to depend on the element of \mathcal{M} , it is better to ask that each element of \mathcal{M} correspond to the same q here. As in Section 7, this can always be arranged by taking suitable powers anyway. If there is a positive lower bound q_0 for the q 's associated to elements of \mathcal{M} , then we can simply treat each element of \mathcal{M} as q_0 -semimetric on X . This uses the fact that a q -semimetric on X is also a q_0 -semimetric when $0 < q_0 \leq q$, as in Section 7.

If there are only finitely many elements of \mathcal{M} , then their maximum is a q -semimetric on X that determines the same topology on X , as in Section 2. If \mathcal{M} is also nondegenerate on X , then the maximum of the elements of \mathcal{M} is a q -metric on X .

Suppose now that \mathcal{M} is countably infinite, and let $d_j(x, y)$ with $j \in \mathbf{Z}_+$ be an enumeration of the elements of \mathcal{M} . Put

$$(25.1) \quad d'_j(x, y) = \min(d_j(x, y), 1/j)$$

for every $x, y \in X$ and $j \in \mathbf{Z}_+$, and

$$(25.2) \quad d(x, y) = \max_{j \geq 1} d'_j(x, y)$$

for every $x, y \in X$. More precisely, $d(x, y) = 0$ when $d'_j(x, y) = 0$ for every j . Otherwise, the maximum in (25.2) can be reduced to the maximum of finitely many positive real numbers, because $d'_j(x, y) \leq 1/j$ for every $x, y \in X$ and $j \geq 1$, by construction. As in the previous section, $d'_j(x, y)$ is a q -semimetric on X that determines the same topology on X as $d_j(x, y)$ for each $j \in \mathbf{Z}_+$. Similarly, one can check that $d(x, y)$ is a q -semimetric on X under these conditions, which is a q -metric on X when \mathcal{M} is nondegenerate on X . We would like to show that

the topology determined on X by $d(x, y)$ is the same as the topology associated to \mathcal{M} .

Observe that

$$(25.3) \quad B_d(x, r) = \bigcap_{j=1}^{\infty} B_{d'_j}(x, r)$$

for every $x \in X$ and $r > 0$, by the definition (25.2) of $d(x, y)$. As usual, the open balls are defined as in (1.5). We also have that

$$(25.4) \quad B_{d'_j}(x, r) = B_{d_j}(x, r)$$

when $r \leq 1/j$, as in (24.2), and that

$$(25.5) \quad B_{d'_j}(x, r) = X$$

when $r > 1/j$, as in (24.3). If $r > 1$, then it follows that (25.3) is equal to X too. Otherwise, if $r \leq 1$, then let $l(r)$ be the largest positive integer such that

$$(25.6) \quad r \leq 1/l(r).$$

Thus (25.3) reduces to

$$(25.7) \quad B_d(x, r) = \bigcap_{j=1}^{l(r)} B_{d_j}(x, r),$$

by (25.4) and (25.5). Using this, one can verify that the topology determined on X by $d(x, y)$ is the same as the one associated to \mathcal{M} , as desired.

If A is a commutative group, then one can apply the previous remarks to translation-invariant q -semimetrics on A , and get translation-invariant q -semimetrics on A as a result. In particular, if A is a commutative topological group with a local base for the topology at 0 with only finitely or countably many elements, then it is well known that there is a translation-invariant semimetric on A that determines the same topology on A . More precisely, if A is any commutative topological group, then there is a collection of translation-invariant semimetrics on A that determines the same topology on A , as mentioned in Section 19. If there is a local base for the topology of A at 0 with only finitely or countably many elements, then only finitely or countably many such semimetrics are needed, which can be reduced to a single semimetric as before. However, a single semimetric is often constructed directly in this case, using the same types of arguments. If $\{0\}$ is a closed set in A , then this semimetric on A is a metric. Of course, if the topology on any set X is determined by a semimetric, then there is a local base for the topology on X at any point $x \in X$ with only finitely or countably many elements, consisting of open balls centered at x with radius $1/j$ for $j \in \mathbf{Z}_+$, for instance.

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a topological vector space over k with respect to $|\cdot|$. In particular, V is a commutative topological group with respect to addition, and so if there is a local base for the topology of V at 0 with only finitely or countably many elements,

then there is a translation-invariant semimetric $d(v, w)$ on V that determines the same topology on V , as in the preceding paragraph. If $|\cdot|$ is nontrivial on k , or if $|\cdot|$ is trivial on V and the topology on V satisfies (20.6), then one can also choose $d(v, w)$ on V so that

$$(25.8) \quad d(tv, tw) \leq d(v, w)$$

for every $v, w \in V$ and $t \in k$ with $|t| \leq 1$. This is typically obtained from the construction of the semimetric, following the analogous construction for commutative topological groups, and using the fact that one can choose a local base for the topology of V at 0 consisting of balanced open sets under these conditions, as in Section 20. One can also get (25.8) as in the previous section.

If the topology on V is determined by a single q -seminorm N , then one can simply use the semimetric associated to N as in (9.2). If the topology on V is determined by finitely many q -seminorms, then one can reduce to the case of a single q -seminorm by taking their maximum. Suppose now that for each $j \in \mathbf{Z}_+$, N_j is a q_j -seminorm on V for some $q_j > 0$, and that the topology on V is determined by the N_j 's. Thus

$$(25.9) \quad N_j(v - w)$$

is a q_j -semimetric on V for each $j \in \mathbf{Z}_+$, as in (9.2). If there is a $q_0 > 0$ such that $q_j \geq q_0$ for each $j \geq 1$, then (25.9) may be considered as a q_0 -semimetric on V for each $j \geq 1$. Otherwise, one can replace (25.9) with

$$(25.10) \quad N_j(v - w)^{q_j}$$

for each j , to get a sequence of semimetrics on V that determines the same topology on V . In both cases, one can get a translation-invariant q_0 -semimetric $d(v, w)$ on V for some $q_0 > 0$ that determines the same topology on V , as in (25.2). More precisely, one would first apply (25.1) to (25.9) or (25.10) for each j , as appropriate, and then define $d(v, w)$ as in (25.2). By construction, $d(v, w)$ also satisfies (25.8), because (25.9) and (25.10) automatically have this property for each j , as in the previous section.

26 Totally bounded sets

Let X be a nonempty set, and let $d(x, y)$ be a q -semimetric on X for some $q > 0$. A set $E \subseteq X$ is said to be *bounded* with respect to d if there is an $x \in X$ and $r > 0$ such that

$$(26.1) \quad E \subseteq B_d(x, r),$$

where $B_d(x, r)$ is as in (1.5). This implies that for each $x \in X$ there is an $r > 0$ such that (26.1) holds, as one can check using the q -semimetric version of the triangle inequality. It follows from this that if E_1, \dots, E_n are finitely many subsets of X that are bounded with respect to d , then their union $\bigcup_{j=1}^n E_j$ is bounded with respect to E as well. If $E \subseteq X$ is compact with respect to the

topology determined on X by d , then it is easy to see that E is bounded with respect to d , by covering E by balls of the form $B_d(x, r)$ for a fixed $x \in X$ and with arbitrarily large r .

A set $E \subseteq X$ is said to be *totally bounded* with respect to d if for every $r > 0$ there are finitely many elements x_1, \dots, x_n of X such that

$$(26.2) \quad E \subseteq \bigcup_{j=1}^n B_d(x_j, r).$$

If this condition holds for any $r > 0$, then E has to be bounded in X with respect to d , because the union of finitely many bounded sets is still bounded, as in the preceding paragraph. Similarly, the union of finitely many subsets of X that are totally bounded with respect to d is totally bounded with respect to d too. If $E \subseteq X$ is compact with respect to the topology determined by d , then E is totally bounded with respect to d , as one can verify by covering E balls of radius r for any $r > 0$. If $E \subseteq X$ is totally bounded with respect to d , then the closure of E with respect to the topology determined on X by d is totally bounded with respect to d as well. If d' is the q -semimetric obtained from d as in (24.1) for some $r_0 > 0$, then d' is automatically bounded on X , so that every subset of X is bounded with respect to d' . However, one can check that $E \subseteq X$ is totally bounded with respect to d' if and only if E is totally bounded with respect to d .

If $E \subseteq X$ is nonempty and bounded with respect to d , then the *diameter* of E with respect to d is defined by

$$(26.3) \quad \text{diam } E = \text{diam}_d E = \sup\{d(x, y) : x, y \in E\}.$$

It is sometimes convenient to take this to be $+\infty$ when E is not bounded with respect to d , and to be 0 when $E = \emptyset$. One can check that $E \subseteq X$ is totally bounded with respect to d if and only if for every $t > 0$, E is contained in the union of finitely many subsets of X with d -diameter less than or equal to t . More precisely, the “if” part of this statement uses the fact that if $A \subseteq X$ has d -diameter less than t , then

$$(26.4) \quad A \subseteq B_d(x, t)$$

for every $x \in A$. The converse uses the fact that any ball in X of radius r with respect to d has d -diameter less than or equal to $2^{1/q}r$, by the q -semimetric version of the triangle inequality.

Let q, \dots, q_n be finitely many positive real numbers, and let d_j be a q_j -semimetric on X for each $j = 1, \dots, n$. Suppose that $E \subseteq X$ is totally bounded with respect to d_j for each $j = 1, \dots, n$, and let t_1, \dots, t_n be positive real numbers. Thus for each $j = 1, \dots, n$, E can be covered by finitely many subsets of X , each of which has d_j -diameter less than or equal to t_j . Using a common refinement of these coverings, one can cover E by finitely many subsets of X , each of which has d_j -diameter less than or equal to t_j for each $j = 1, \dots, n$ simultaneously.

27 Totally bounded sets, continued

Let A be a commutative topological group. A set $E \subseteq A$ is said to be *totally bounded* in A if for each open set $U \subseteq A$ with $0 \in U$ there are finitely many elements x_1, \dots, x_n of A such that

$$(27.1) \quad E \subseteq \bigcup_{j=1}^n (x_j + U).$$

As before, it is easy to see that the union of finitely many totally bounded subsets of A is also totally bounded, and that compact subsets of A are totally bounded. If $E \subseteq A$ is totally bounded, then every translate of E in A is totally bounded too, as is $-E$. If \mathcal{B}_0 is a local base for the topology of A at 0, then it suffices to show that $E \subseteq A$ can be covered by finitely many translates of each $U \in \mathcal{B}_0$, in order to verify that E is totally bounded.

Let W be an open subset of A that contains 0, and let us say that a set $C \subseteq A$ is *W -small* if

$$(27.2) \quad C - C \subseteq W.$$

Equivalently, this means that $x - y \in W$ for every $x, y \in C$, which is the same as saying that

$$(27.3) \quad C \subseteq y + W$$

for every $y \in C$. Of course, if C is W -small, then every subset of C is W -small as well. Put

$$(27.4) \quad \widetilde{W} = W - W,$$

which is also an open subset of A that contains 0. If (27.3) holds for any $y \in A$, then

$$(27.5) \quad C - C \subseteq W - W = \widetilde{W},$$

so that C is \widetilde{W} -small in A .

Let $U \subseteq A$ be an open set that contains 0 and satisfies

$$(27.6) \quad U - U \subseteq W,$$

which exists by the continuity of the group operations on A at 0. If $E \subseteq A$ is totally bounded, then E can be covered by finitely many translates of U in A , as in (27.1). Each translate of U in A is W -small in A , so that E can be covered by finitely many sets which are W -small in A . Conversely, if E can be covered by finitely many subsets of A that are W -small, then E can be covered by finitely many translates of W . Thus E is totally bounded in A if and only if for every open set $W \subseteq A$ that contains 0, E can be covered by finitely many subsets of A that are W -small.

Let B be a subgroup of A , which is also a commutative topological group with respect to the topology induced by the one on A . If $W \subseteq A$ is an open set that contains 0, then

$$(27.7) \quad W_B = B \cap W$$

is a relatively open set in B that contains 0. It is easy to see that $C \subseteq B$ is W -small in A if and only if C is W_B -small in B . If $C \subseteq A$ is W -small, then $B \cap C$ is W -small in A too, which implies that $B \cap C$ is W_B -small in B . This implies that $E \subseteq B$ can be covered by finitely many subsets of B that are W_B -small if and only if E can be covered by finitely many subsets of A that are W -small. It follows that $E \subseteq B$ is totally bounded in B if and only if E is totally bounded in A . This also uses the fact that every relatively open subset of B that contains 0 is of the form W_B for some open set $W \subseteq A$ that contains 0.

Let W_1, \dots, W_n be finitely many open subsets of A that contain 0, so that

$$(27.8) \quad W = \bigcap_{j=1}^n W_j$$

is an open set that contains 0 too. Let $E \subseteq A$ be given, and suppose that for each $j = 1, \dots, n$, E can be covered by finitely many W_j -small subsets of A . Using a common refinement of these coverings, one can cover E by finitely many W -small subsets of A . Let \mathcal{B}_0 be a local sub-base for the topology of A at 0, so that every open set in A that contains 0 also contains the intersection of finitely many elements of \mathcal{B}_0 . If for each $U \in \mathcal{B}_0$, E can be covered by finitely many U -small subsets of A , then it follows that E is totally bounded in A .

Suppose for the moment that the topology on A is determined by a collection \mathcal{M} of translation-invariant q -semimetrics on A , where $q > 0$ is allowed to depend on the element of \mathcal{M} . If $E \subseteq A$ is totally bounded as a subset of A as a commutative topological group, then it is easy to see that E is totally bounded with respect to every element of \mathcal{M} . This uses the fact that open balls in A with respect to elements of \mathcal{M} are open sets in A , and that

$$(27.9) \quad B_d(x, r) = x + B_d(0, r)$$

for every $x \in A$, $r > 0$, and $d \in \mathcal{M}$, because d is supposed to be invariant under translations on A . Conversely, if $E \subseteq A$ is totally bounded with respect to every element of \mathcal{M} , then one can check that E is totally bounded in A as a commutative topological group. More precisely, let d_1, \dots, d_n be finitely many elements of \mathcal{M} , and let r_1, \dots, r_n be finitely many positive real numbers. Thus

$$(27.10) \quad U = \bigcap_{j=1}^n B_{d_j}(0, r_j)$$

is an open set in A that contains 0, and subsets of A of this form determine a local base for the topology of A at 0, as in Section 2. If $E \subseteq A$ is totally bounded with respect to d_j for each $j = 1, \dots, n$, then E can be covered by finitely many subsets of X , each of which has d_j -diameter less than r_j for every $j = 1, \dots, n$, as in the preceding section. This implies that E can be covered by finitely many translates of (27.10) in A , as desired. Alternatively, this can be derived from the remarks in paragraph, since the collection of open balls in A centered at 0 with respect to elements of \mathcal{M} form a local sub-base for the topology of A at 0.

Suppose that $E \subseteq A$ is totally bounded in A as a commutative topological group, and let $W \subseteq A$ be an open set that contains 0. Thus there are open sets $U_1, U_2 \subseteq A$ that contain 0 and satisfy

$$(27.11) \quad U_1 + U_2 \subseteq W,$$

by continuity of addition on A at 0. Because E is totally bounded in A , there are finitely many elements x_1, \dots, x_n of A such that

$$(27.12) \quad E \subseteq \bigcup_{j=1}^n (x_j + U_1).$$

This implies that

$$(27.13) \quad \overline{E} \subseteq E + U_2 \subseteq \bigcup_{j=1}^n (x_j + U_1 + U_2) \subseteq \bigcup_{j=1}^n (x_j + W),$$

where \overline{E} is the closure of E in A , and using (18.8) in the first step. It follows that \overline{E} is totally bounded in A too. Alternatively, let $U \subseteq A$ be an open set that contains 0 and satisfies

$$(27.14) \quad \overline{U} \subseteq W,$$

as in Section 18. If E is totally bounded, then E can be covered by finitely many translates of U , which implies that \overline{E} can be covered by finitely many translates of \overline{U} , and hence by finitely many translates of W .

Suppose that $E_1, E_2 \subseteq A$ are totally bounded, and let $W \subseteq A$ be an open set that contains 0 again. Thus there are open sets $U_1, U_2 \subseteq A$ that contain 0 and satisfy (27.11), as before. Because E_1 and E_2 are totally bounded, there are finitely many elements x_1, \dots, x_m and y_1, \dots, y_n of A such that

$$(27.15) \quad E_1 \subseteq \bigcup_{j=1}^m (x_j + U_1) \quad \text{and} \quad E_2 \subseteq \bigcup_{l=1}^n (y_l + U_2).$$

This implies that

$$(27.16) \quad E_1 + E_2 \subseteq \bigcup_{j=1}^m \bigcup_{l=1}^n (x_j + y_l + U_1 + U_2) \subseteq \bigcup_{j=1}^m \bigcup_{l=1}^n (x_j + y_l + W),$$

using (27.11) in the second step. It follows that $E_1 + E_2$ is totally bounded in A as well.

Let I be a nonempty set, and suppose that A_j is a commutative topological group for each $j \in I$. As in Section 23,

$$(27.17) \quad A = \prod_{j \in I} A_j$$

is also a commutative topological group, where the group operations are defined coordinatewise, and using the product topology on A associated to the given

topology on A_j for each $j \in I$. If $E_j \subseteq A_j$ is totally bounded for each $j \in I$, then one can check that

$$(27.18) \quad E = \prod_{j \in I} E_j$$

is totally bounded in A , with respect to the product topology on A . More precisely, if $U_j \subseteq A_j$ is an open set that contains 0 for each $j \in I$, and if $U_j = A_j$ for all but finitely many $j \in I$, then

$$(27.19) \quad U = \prod_{j \in I} U_j$$

is an open set in A with respect to the product topology that contains 0. In addition, the collection of subsets of A of this form is a local base for the product topology on A at 0. In order to show that $E \subseteq A$ is totally bounded, it suffices to verify that E can be covered by finitely many translates of U in A , for each $U \subseteq A$ of this form. If $E_j \subseteq A_j$ is totally bounded, then E_j can be covered by finitely many translates of U_j in A_j . If E is as in (27.18), then one can cover E by finitely many translates of U in A , using the coverings of E_j by finitely many translates of U_j in A_j for each of the finitely many $j \in I$ such that $U_j \neq A_j$.

28 Bounded sets

Let k be a field, and let $|\cdot|$ be a nontrivial q -absolute value function on k for some $q > 0$. Also let V be a topological vector space over k , with respect to $|\cdot|$ on k . A set $E \subseteq V$ is said to be *bounded* if for each open set $U \subseteq V$ that contains 0 there is a $t_0 \in k$ such that

$$(28.1) \quad E \subseteq t_0 U.$$

If U is balanced in V , then it follows that

$$(28.2) \quad E \subseteq t U$$

for every $t \in k$ with $|t| \geq |t_0|$. If U is any open set in V that contains 0, then there is a balanced open set $U_1 \subseteq A$ such that $0 \in U_1$ and $U_1 \subseteq U$, as in Section 20. If $E \subseteq V$ is a bounded set, then there is a $t_1 \in k$ such that

$$(28.3) \quad E \subseteq t_1 U_1,$$

as in (28.1). This implies that

$$(28.4) \quad E \subseteq t U_1$$

for every $t \in k$ with $|t| \geq |t_1|$, as in (28.2), because U_1 is balanced in V . It follows that (28.2) holds for every $t \in k$ with $|t| \geq |t_1|$, because $U_1 \subseteq U$.

Let \mathcal{B}_0 be a local base for the topology of V at 0. If $E \subseteq V$ has the property that for each $U \in \mathcal{B}_0$ there is a $t_0 \in k$ that satisfies (28.1), then it is easy to see that E is bounded in V . In particular, one can take \mathcal{B}_0 to be the collection

of all balanced open sets in V , which basically corresponds to the discussion in the previous paragraph.

Remember that open subsets of V that contain 0 are absorbing, as in Section 20. This implies that subsets of V with only one element are bounded, and in fact that finite subsets of V are bounded. Similarly, one can check that the union of finitely many bounded subsets of V is bounded, using the characterization of boundedness in terms of (28.2) holding when $|t|$ is sufficiently large.

Let $U \subseteq V$ be an open set that contains 0. If $\{t_j\}_{j=1}^{\infty}$ is a sequence of elements of k such that $|t_j| \rightarrow \infty$ as $j \rightarrow \infty$, then

$$(28.5) \quad \bigcup_{j=1}^{\infty} t_j U = V,$$

because U is absorbing in V . If U is balanced in V , and if $|t_j| \leq |t_{j+1}|$ for every j , then we get that

$$(28.6) \quad t_j U \subseteq t_{j+1} U$$

for each j . Let us also ask that $t_j \neq 0$ for each j , so that $t_j U$ is an open set in V for every j . In particular, if $t \in k$ and $|t| > 1$, then these conditions hold with $t_j = t^j$. If $E \subseteq V$ is compact, then (28.5) and (28.6) imply that

$$(28.7) \quad E \subseteq t_j U$$

for some j . This implies that E is bounded in V , since we can restrict our attention to balanced open subsets U of V in the definition of boundedness.

Suppose now that $E \subseteq V$ is totally bounded, as a subset of V as a commutative topological group with respect to addition. Also let $W \subseteq V$ be an open set that contains 0, and let $U_1, U_2 \subseteq V$ be open sets that contain 0 and satisfy

$$(28.8) \quad U_1 + U_2 \subseteq W,$$

as in (18.11). We may as well ask that U_1, U_2 be balanced in V too, since otherwise they can be replaced by balanced open subsets, as in Section 20. Because E is totally bounded in V , there are finitely many elements v_1, \dots, v_n of V such that

$$(28.9) \quad E \subseteq \bigcup_{j=1}^n (v_j + U_2),$$

as in (27.1). This implies that

$$(28.10) \quad E \subseteq t U_1 + U_2$$

for every $t \in k$ such that $|t|$ is sufficiently large, because U_1 is absorbing in V . Note that $U_2 \subseteq t U_2$ when $|t| \geq 1$, since U_2 is balanced in V . It follows that

$$(28.11) \quad E \subseteq t U_1 + t U_2 \subseteq t W$$

when $|t|$ is sufficiently large, using (28.8) in the second step. This shows that totally bounded subsets of V are bounded in V .

Let $E \subseteq V$ be a bounded set, let $W \subseteq V$ be an open set that contains 0, and let $U_1, U_2 \subseteq V$ be open sets that contain 0 and satisfy (28.8). Thus

$$(28.12) \quad E \subseteq tU_1$$

for every $t \in k$ such that $|t|$ is sufficiently large, because E is bounded in V . We also have that

$$(28.13) \quad \overline{E} \subseteq E + tU_2$$

for every $t \in k$ with $t \neq 0$, where \overline{E} is the closure of E in V . This follows from (18.8) and the fact that tU_2 is an open set in V when $t \neq 0$. Combining (28.12) and (28.13), we get that

$$(28.14) \quad \overline{E} \subseteq tU_1 + tU_2 = tW$$

for every $t \in k$ such that $|t|$ is sufficiently large, which implies that \overline{E} is bounded in V too.

Let $E_1, E_2 \subseteq V$ be bounded sets, and let us check that $E_1 + E_2$ is also a bounded subset of V . To do this, let $W \subseteq V$ be an open set that contains 0, and let $U_1, U_2 \subseteq V$ be open sets that contain 0 and satisfy (28.8). Thus

$$(28.15) \quad E_1 \subseteq tU_1 \quad \text{and} \quad E_2 \subseteq tU_2$$

when $t \in k$ and $|t|$ is sufficiently large, which implies that

$$(28.16) \quad E_1 + E_2 \subseteq tU_1 + tU_2 = tW$$

when $|t|$ is sufficiently large, as desired.

Let I be a nonempty set, and let V_j be a topological vector space over k for each $j \in I$. Remember that

$$(28.17) \quad V = \prod_{j \in I} V_j$$

is a topological vector space over k as well, as in Section 23, where the vector space operations are defined coordinatewise, and using the product topology on V associated to the given topology on V_j for each $j \in I$. Suppose that $E_j \subseteq V_j$ is a bounded set for each $j \in I$, and let us check that

$$(28.18) \quad E = \prod_{j \in I} E_j$$

is bounded in V . To do this, let $U_j \subseteq V_j$ be an open set that contains 0 for each $j \in I$, and suppose that $U_j = V_j$ for all but finitely many $j \in I$. Thus

$$(28.19) \quad U = \prod_{j \in I} U_j$$

is an open set in V with respect to the product topology that contains 0, and the collection of subsets of V of this form is a local base for the topology of V at 0. In order to show that E is bounded in V , it suffices to verify that

$$(28.20) \quad E \subseteq tU$$

for every $t \in k$ such that $|t|$ is sufficiently large. Of course, for each $j \in I$, we know that

$$(28.21) \quad E_j \subseteq t U_j$$

for every $t \in k$ such that $|t|$ is sufficiently large, because E_j is bounded in V_j . By hypothesis, $U_j = V_j$ for all but finitely many $j \in I$, in which case (28.21) holds for every $t \in k$ with $t \neq 0$. If $|t|$ is sufficiently large, then it follows that (28.21) holds for all $j \in I$ simultaneously, which implies that (28.20) holds, as desired.

Let V be any topological vector space over k again, and let N be a nonnegative real-valued function on V that satisfies the usual homogeneity condition (6.1). If N is continuous at 0, then there is an open set $U \subseteq V$ that contains 0 and satisfies

$$(28.22) \quad U \subseteq B_N(0, 1),$$

where $B_N(0, r)$ is as in (10.7). Conversely, this condition implies that N is continuous at 0, because of (10.9), and our standing hypothesis in this section that $|\cdot|$ be nontrivial on k . If $E \subseteq V$ is bounded, then it follows that N is bounded on E in this situation. If N is a q -seminorm on V for some $q > 0$, then continuity of N at 0 on V implies that N is continuous on V , as in Section 21. In particular, this implies that open balls in V with respect to N are open sets in V , as before. In this case, the boundedness of N on E is the same as the boundedness of E with respect to the corresponding q -semimetric (9.2), as in the previous section.

Now let V be a vector space over k , and let \mathcal{N} be a collection of q -seminorms on V . As usual, we can let q depend on the element of \mathcal{N} , as long as $|\cdot|$ is a q -absolute value function on k . As in Section 21, V is a topological vector space with respect to the topology associated to \mathcal{N} . If $E \subseteq V$ is bounded with respect to the topology associated to \mathcal{N} , then each element of \mathcal{N} is bounded on E . This follows from the remarks in the preceding paragraph, since every element of \mathcal{N} is continuous with respect to the corresponding topology on V . Conversely, suppose that $E \subseteq V$ has the property that every element of \mathcal{N} is bounded on E . Let $U \subseteq V$ be an open set with respect to the topology associated to \mathcal{M} that contains 0. By construction, this means that there are finitely many elements N_1, \dots, N_l of \mathcal{N} and finitely many positive real numbers r_1, \dots, r_l such that

$$(28.23) \quad \bigcap_{j=1}^l B_{N_j}(0, r_j) \subseteq U.$$

Using this, it is easy to see that $E \subseteq t U$ for every $t \in k$ such that $|t|$ is sufficiently large, because N_j is bounded on E for each $j = 1, \dots, l$. This implies that E is bounded with respect to the topology on V associated to \mathcal{N} .

Let V be any topological vector space over k , and suppose that $E \subseteq V$ is bounded. If $\{v_j\}_{j=1}^\infty$ is a sequence of elements of E , and if $\{t_j\}_{j=1}^\infty$ is a sequence of elements of k that converges to 0 with respect to $|\cdot|$, then it is easy to see that $\{t_j v_j\}_{j=1}^\infty$ converges to 0 in V . Conversely, suppose that E is not bounded in V , so that there is an open set $U \subseteq V$ such that $0 \in U$ and $E \not\subseteq t U$ for every

$t \in k$. Let t_0 be an element of k such that $|t_0| > 1$, which exists because $|\cdot|$ is nontrivial on k . Thus $E \not\subseteq t_0^j U$ for every positive integer j , and so we can choose $v_j \in E \setminus (t_0^j U)$ for each j . This implies that $t_0^{-j} v_j \notin U$ for each j , so that $\{t_0^{-j} v_j\}_{j=1}^\infty$ does not converge to 0 in V . Of course, $\{t_0^{-j}\}_{j=1}^\infty$ does converge to 0 in k , since $|t_0| > 1$.

29 Continuous functions

If X and Y are topological spaces, then we let $C(X, Y)$ be the space of continuous mappings from X into Y . Let X be a nonempty topological space, and let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$. This leads to a topology on k in the usual way, corresponding to the q -metric (8.2) associated to $|\cdot|$ on k . Note that $C(X, k)$ is a vector space over k with respect to pointwise addition and scalar multiplication. More precisely, $C(X, k)$ is a commutative algebra over k with respect to pointwise multiplication of functions.

Let E be a nonempty compact subset of X , and let f be a continuous k -valued function on X . Thus $f(E)$ is a nonempty compact subset of k , which is bounded with respect to $|\cdot|$ in particular. This permits us to put

$$(29.1) \quad \|f\|_E = \sup_{x \in E} |f(x)|,$$

since the right side of (29.1) is the supremum of a nonempty set of nonnegative real numbers. Remember that $|\cdot|$ is continuous as a real-valued function on k with respect to the topology determined on k by the corresponding q -metric, as in Section 17. This implies that $|f(x)|$ is continuous as a real-valued function on X , so that the supremum in the right side of (29.1) is attained.

It is easy to see that (29.1) defines a q -seminorm on $C(X, k)$ for every nonempty compact set $E \subseteq X$. This is the *supremum q -seminorm* associated to E . If X is compact, then we can take $E = X$ in (29.1), which defines a q -norm on $C(X, k)$. We also have that

$$(29.2) \quad \|fg\|_E \leq \|f\|_E \|g\|_E$$

for every $f, g \in C(X, k)$ and nonempty compact set $E \subseteq X$, because of the multiplicative property of absolute value functions.

Let \mathcal{N} be the collection of supremum q -seminorms on $C(X, k)$ that are associated to nonempty compact subsets E of X as in (29.1). This is a nondegenerate collection of q -seminorms on $C(X, k)$, because finite subsets of X are compact. As in Section 21, $C(X, k)$ is a Hausdorff topological vector space over k with respect to the topology determined by \mathcal{N} . Similarly, one can check that multiplication of functions defines a continuous mapping from $C(X, k) \times C(X, k)$ into $C(X, k)$, where $C(X, k) \times C(X, k)$ is equipped with the product topology associated to the topology just defined on $C(X, k)$, using (29.2). If X is compact, then the same topology on $C(X, k)$ is determined by the supremum q -norm $\|f\|_X$.

Observe that $C(X, k \setminus \{0\})$ is a commutative group with respect to multiplication. In fact, $C(X, k \setminus \{0\})$ is a commutative topological group, with respect

to the topology induced by the one on $C(X, k)$ described in the preceding paragraph. Continuity of multiplication on $C(X, k \setminus \{0\})$ follows from the analogous statement for $C(X, k)$ just mentioned. The remaining point is that $f \mapsto 1/f$ defines a continuous mapping on $C(X, k \setminus \{0\})$ with respect to the induced topology, which can be verified using standard arguments. In particular, if f is a continuous function on X with values in $k \setminus \{0\}$, then $1/f$ is bounded on compact subsets of X , since it is continuous on X too.

Put

$$(29.3) \quad \mathbf{T}_k = \{x \in k : |x| = 1\},$$

which is a subgroup of $k \setminus \{0\}$ with respect to multiplication. Thus $C(X, \mathbf{T}_k)$ is a subgroup of $C(X, k \setminus \{0\})$ with respect to multiplication of functions. It follows that $C(X, \mathbf{T}_k)$ is also a commutative topological group with respect to the topology induced by the one on $C(X, k)$ mentioned earlier. Note that \mathbf{T}_k is a closed set in k with respect to the topology determined by the q -metric associated to $|\cdot|$. Using this, one can check that $C(X, \mathbf{T}_k)$ is a closed set in $C(X, k)$, with respect to the usual topology.

Let E be a nonempty compact subset of X , and observe that

$$(29.4) \quad \|a f\|_E = \|f\|_E$$

for every $a \in C(X, \mathbf{T}_k)$ and $f \in C(X, k)$. If

$$(29.5) \quad d_E(f, g) = \|f - g\|_E$$

is the q -semimetric associated to $\|\cdot\|_E$, then we get that

$$(29.6) \quad d_E(a f, a g) = d_E(f, g)$$

for every $a \in C(X, \mathbf{T}_k)$ and $f, g \in C(X, k)$. The restriction of (29.5) to f, g in $C(X, \mathbf{T}_k)$ defines a q -semimetric on $C(X, \mathbf{T}_k)$, and (29.6) implies that this semimetric is invariant under translations on $C(X, \mathbf{T}_k)$, as a group with respect to multiplication. By construction, the usual topology on $C(X, k)$ is the one determined by the collection of q -semimetrics (29.5) associated to nonempty compact sets $E \subseteq X$. The induced topology on $C(X, \mathbf{T}_k)$ is the same as the one determined by the collection of restrictions of these q -semimetrics to $C(X, \mathbf{T}_k)$, as in Section 2.

Suppose for the moment that X is any nonempty set equipped with the discrete topology. This implies that every function on X is continuous, and that every compact subset of X has only finitely many elements. In this case, $C(X, k)$ can be identified with the Cartesian product of a family of copies of k indexed by X , as in Section 23. The topology on $C(X, k)$ described earlier corresponds exactly to the product topology in this situation, using the topology on each factor of k determined by the q -metric associated to $|\cdot|$. Similarly, $C(X, k \setminus \{0\})$ and $C(X, \mathbf{T}_k)$ can be identified with Cartesian products of copies of $k \setminus \{0\}$ and \mathbf{T}_k indexed by X , with their appropriate product topologies.

Suppose now that X is a nonempty topological space which is locally compact, in the sense that every point in X is contained in an open subset of X that

is contained in a compact subset of X . This implies that every compact subset of X is contained in an open subset of X that is contained in a compact subset of X , by standard arguments. Suppose also that X is σ -compact, which means that there is a sequence E_1, E_2, E_3, \dots of compact subsets of X such that

$$(29.7) \quad \bigcup_{j=1}^{\infty} E_j = X.$$

We may as well suppose that $E_j \subseteq E_{j+1}$ for each j , since otherwise we can replace E_j with $\bigcup_{l=1}^j E_l$ for each j . Using local compactness, we can refine this a bit, to get that for each $j \geq 1$ there is an open set $U_j \subseteq X$ such that

$$(29.8) \quad E_j \subseteq U_j \subseteq E_{j+1}.$$

More precisely, if E_j has already been chosen for some j , then we take U_j to be an open set in X that contains E_j and is contained in a compact subset of X . In the next step, we expand E_{j+1} so that it contains U_j , and continue as before.

It follows that $U_j \subseteq U_{j+1}$ for each j , and that

$$(29.9) \quad \bigcup_{j=1}^{\infty} U_j = X.$$

If E is any compact subset of X , then E is contained in the union of finitely many U_j 's, and hence in a single U_j . This implies that

$$(29.10) \quad E \subseteq E_j$$

for some j . We may as well choose E_1 to be nonempty, so that $E_j \neq \emptyset$ for every j , which means that the supremum q -seminorm associated to E_j may be defined on $C(X, k)$ for each j as before. Under these conditions, the sequence of supremum q -seminorms associated to the E_j 's are sufficient to define the usual topology on $C(X, k)$.

30 Continuous linear mappings

Let X_1 and X_2 be topological spaces, and let ϕ be a mapping from X_1 into X_2 . As usual, ϕ is said to be *sequentially continuous* at a point $x \in X_1$ if for each sequence $\{x_j\}_{j=1}^{\infty}$ of elements of X_1 that converges to x , $\{\phi(x_j)\}_{j=1}^{\infty}$ converges to $\phi(x)$ in X_2 . Of course, if ϕ is continuous at x , then ϕ is sequentially continuous at x . If there is a local base for the topology of X_1 at x with only finitely or countably many elements, and if ϕ is sequentially continuous at x , then it is well known that ϕ is continuous at x . To see this, suppose for the sake of a contradiction that there is an open set $W \subseteq X_2$ such that $\phi(x) \in W$, but

$$(30.1) \quad \phi(U) \not\subseteq W$$

for every open set $U \subseteq X_1$ that contains x . The hypothesis that there be a local base for the topology of X_1 at x with only finitely or countably many elements means that there is a sequence $U_1(x), U_2(x), U_3(x), \dots$ of open subsets of X_1 that contain x such that if U is any other open subset of X_1 that contains x , then $U_l(x) \subseteq U$ for some $l \geq 1$. We may as well ask also that

$$(30.2) \quad U_{l+1}(x) \subseteq U_l(x)$$

for every l , since otherwise we can replace $U_l(x)$ with $\bigcap_{j=1}^l U_j(x)$ for each l . Our hypothesis (30.1) implies that for each positive integer l there is an $x_l \in U_l(x)$ such that

$$(30.3) \quad \phi(x_l) \notin W.$$

Under these conditions, $\{x_l\}_{l=1}^\infty$ converges to x in X_1 , but $\{\phi(x_l)\}_{l=1}^\infty$ does not converge to $\phi(x)$ in X_2 , as desired.

Now let A_1, A_2 be commutative topological groups, and let ϕ be a group homomorphism from A_1 into A_2 . If ϕ is continuous at 0, then for each open set $W \subseteq A_2$ that contains 0 there is an open set $U \subseteq A_1$ that contains 0 such that

$$(30.4) \quad \phi(U) \subseteq W.$$

This implies that

$$(30.5) \quad \phi(x + U) \subseteq \phi(x) + W$$

for every $x \in A_1$, and hence that ϕ is continuous at every point in A_1 . More precisely, this implies that ϕ is uniformly continuous as a mapping from A_1 into A_2 in a suitable sense. In particular, if $E \subseteq A_1$ is totally bounded, then one can use this to check that $\phi(E)$ is totally bounded in A_2 . If ϕ is sequentially continuous at 0, then it is easy to see that ϕ is sequentially continuous at every point in A_1 . If there is also a local base for the topology of A_1 at 0 with only finitely or countably many elements, then ϕ is continuous at 0, as in the preceding paragraph.

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Remember that topological vector spaces over k are commutative topological groups with respect to addition, and that linear mappings between vector space over k are group homomorphisms with respect to addition. Thus the remarks in the previous paragraph can be applied to linear mappings between topological vector spaces over k . Note that k may be considered as a one-dimensional topological vector space over itself, using the topology determined on k by the q -metric associated to $|\cdot|$.

Let V be a topological vector space over k , and let $v \in V$ be given. Put

$$(30.6) \quad \phi_v(t) = t v$$

for each $t \in k$, which defines a continuous linear mapping from k into V , because of continuity of scalar multiplication on V , as in Section 20. Suppose that $v \neq 0$, and put

$$(30.7) \quad L_v = \phi_v(k) = \{t v : t \in k\},$$

which is the one-dimensional linear subspace of V spanned by v . Let us ask also that $\{0\}$ be a closed set in V , so that $\{v\}$ is a closed set in V too, by continuity of translations. Thus $V \setminus \{v\}$ is an open set in V that contains 0.

If $|\cdot|$ is nontrivial on k , then there is a balanced open set $U \subseteq V$ contained in $V \setminus \{v\}$, as in Section 20. Because U is balanced, we have that

$$(30.8) \quad tv \notin U$$

for every $t \in k$ with $|t| \geq 1$. If $a \in k$ and $a \neq 0$, then aU is also a balanced open set in V that contains 0, and

$$(30.9) \quad tv \notin aU$$

for every $t \in k$ with $|t| \geq |a|$, by (30.8). Using this, one can check that ϕ_v is a homeomorphism from k onto L_v under these conditions, with respect to the topology induced on L_v by the one on V . More precisely, (30.9) implies that the inverse of ϕ_v is continuous at 0 as a mapping from L_v into k .

Let us ask that $|\cdot|$ be nontrivial on k for the rest of the section, and let V_1 and V_2 be topological vector spaces over k . A linear mapping ϕ from V_1 into V_2 is said to be *bounded* if for each bounded set $E \subseteq V_1$, we have that $\phi(E)$ is a bounded subset of V_2 , where boundedness is defined as in Section 28. If ϕ is continuous, then it is easy to see that ϕ is bounded, directly from the definitions.

Suppose for the moment that U is an open subset of V_1 that contains 0. If $\phi(U)$ is a bounded subset of V_2 , then one can check that ϕ is continuous. In particular, this holds when U is also bounded in V_1 , and ϕ is a bounded linear mapping. If there is a bounded open set $W \subseteq V_2$ that contains 0, and if ϕ is continuous, then there is an open set $U \subseteq V_1$ that contains 0 such that $\phi(U) \subseteq W$, which implies that $\phi(U)$ is bounded in V_2 . If the topology on a vector space V over k is determined by a single q -seminorm N , then open balls with respect to N are both bounded and open in V .

Suppose now that there is a local base for the topology of V_1 at 0 with only finitely or countably many elements. Thus there is a sequence U_1, U_2, U_3, \dots of open subsets of V_1 that contain 0 such that any other open subset of V_1 that contains 0 also contains U_l for some l . We may as well take U_l to be balanced in V_1 for each l , because $|\cdot|$ is nontrivial on k , as in Section 20. We may also ask that

$$(30.10) \quad U_{l+1} \subseteq U_l$$

for each l , since otherwise we can replace U_l with $\bigcap_{j=1}^l U_j$ for each l , as before. Note that U_l is absorbing in V_1 for each l , as in Section 20, and using the nontriviality of $|\cdot|$ on V again. This permits us to define

$$(30.11) \quad N_l(v) = N_{U_l}(v)$$

for each $v \in V_1$ and $l \geq 1$ as in (14.1). Using (30.10), we get that

$$(30.12) \quad N_l(v) \leq N_{l+1}(v)$$

for every $v \in V_1$ and $l \geq 1$, as in (16.2).

Let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of V_1 . If $\{v_j\}_{j=1}^\infty$ converges to 0 in V_1 , then

$$(30.13) \quad \lim_{j \rightarrow \infty} N_l(v_j) = 0$$

for every $l \geq 1$, because tU_l is an open set in V that contains 0 for each $l \geq 1$ and $t \in k$ with $t \neq 0$. Conversely, if (30.13) holds for every l , then $\{v_j\}_{j=1}^\infty$ converges to 0 in V_1 , because the U_l 's determine a local base for the topology of V_1 at 0. In fact, it suffices that the U_l 's and their dilates by nonzero elements of k determine a local base for the topology of V_1 at 0 for this to work.

If $\{l_j\}_{j=1}^\infty$ is a sequence of positive integers such that

$$(30.14) \quad l_j \rightarrow \infty \quad \text{as } j \rightarrow \infty$$

and

$$(30.15) \quad \lim_{j \rightarrow \infty} N_{l_j}(v_j) = 0,$$

then (30.13) holds for each l , because of (30.12). This implies that $\{v_j\}_{j=1}^\infty$ converges to 0 in V_1 , as before. Conversely, if $\{v_j\}_{j=1}^\infty$ converges to 0 in V_1 , then there is a sequence $\{l_j\}_{j=1}^\infty$ of positive integers that satisfies (30.14) and (30.15). Of course, this uses the fact that (30.13) holds for each l .

If $\{v_j\}_{j=1}^\infty$ converges to 0 in V_1 , then there is a sequence $\{t_j\}_{j=1}^\infty$ of nonzero elements of k that converges to 0 in k and has the property that

$$(30.16) \quad \lim_{j \rightarrow \infty} t_j^{-1} v_j = 0$$

in V_1 . More precisely, if $\{l_j\}_{j=1}^\infty$ is a sequence of positive integers that satisfies (30.14) and (30.15), then it suffices to show that there is a sequence $\{t_j\}_{j=1}^\infty$ of nonzero elements of k that converges to 0 and satisfies

$$(30.17) \quad N_{l_j}(t_j^{-1} v_j) = |t_j|^{-1} N_{l_j}(v_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

In order to get such a sequence $\{t_j\}_{j=1}^\infty$, one can use the nontriviality of $|\cdot|$ on k , which implies that there are elements of k whose absolute value is comparable to any given positive real number.

Suppose that ϕ is a bounded linear mapping from V_1 into V_2 , and let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of V_1 that converges to 0. As in the preceding paragraph, there is a sequence $\{t_j\}_{j=1}^\infty$ of nonzero elements of k that converges to 0 and satisfies (30.16). This implies that

$$(30.18) \quad E = \{t_j^{-1} v_j : j \in \mathbf{Z}_+\} \cup \{0\}$$

is a compact subset of V_1 , by standard arguments, and hence that E is bounded in V_1 , as in Section 28. It follows that $\phi(E)$ is a bounded set in V_2 , by hypothesis, so that

$$(30.19) \quad \phi(v_j) = t_j \phi(t_j^{-1} v_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

in V_2 , because $\{t_j\}_{j=1}^\infty$ converges to 0 in k . This criterion for convergence of sequences using bounded sets was mentioned in Section 28. This shows that

ϕ is sequentially continuous at 0 under these conditions. Thus ϕ is continuous on V_1 , since V_1 is supposed to have a local base for its topology at 0 with only finitely or countably many elements.

31 The strong product topology

Let I be a nonempty set, and let X_j be a topological space for each $j \in I$. As in Section 23, we let

$$(31.1) \quad X = \prod_{j \in I} X_j$$

be the Cartesian product of the X_j 's, and we let $x_j \in X_j$ be the j th coordinate of $x \in X$ for each $j \in I$. A set $W \subseteq X$ is said to be an open set with respect to the *strong product topology* if for each $x \in W$ and $j \in I$ there is an open set $U_j \subseteq X_j$ such that $x_j \in U_j$ and

$$(31.2) \quad U = \prod_{j \in I} U_j$$

is contained in W . It is well known and not difficult to check that this defines a topology on X , which is the same as the product topology on X when I has only finitely many elements. If I is any nonempty set, then every open set in X with respect to the product topology is also an open set with respect to the strong product topology. Equivalently, if $U_j \subseteq X_j$ is an open set for each $j \in I$, then (31.2) is an open set in X with respect to the strong product topology, and the collection of these open sets forms a base for the strong product topology on X . If X_j is equipped with the discrete topology for each $j \in I$, then the strong product topology on X is the same as the discrete topology on X .

Now let A_j be a commutative topological group for each $j \in I$, and let

$$(31.3) \quad A = \prod_{j \in I} A_j$$

be their Cartesian product. As in Section 23, A is also a commutative group with respect to coordinatewise addition. One can check that A is a topological group with respect to the strong product topology as well.

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V_j be a topological vector space over k for each $j \in I$. As in Section 23, the Cartesian product

$$(31.4) \quad V = \prod_{j \in I} V_j$$

is a vector space over k too, where the vector space operations are defined coordinatewise. In particular, V_j is a commutative topological group with respect to addition for each j , so that V is also a commutative topological group with respect to addition, as in the preceding paragraph. It is easy to see that for each $t \in k$, multiplication by t is continuous on V with respect to the strong product

topology, because of the analogous property of V_j for each $j \in I$. However, if I has infinitely many elements, then V is not necessarily a topological vector space over k with respect to the strong product topology. More precisely, if $v \in V$ has infinitely many nonzero coordinates, then $t \mapsto t v$ is not necessarily continuous as a mapping from k into V , with respect to the strong product topology on V . Of course, this is not a problem when $|\cdot|$ is the trivial absolute value function on k .

If $|\cdot|$ is nontrivial on k , then for each $j \in I$, there is a local base for the topology of V_j at 0 consisting of balanced open sets in V_j , as in Section 20. This leads to a local base for the strong product topology on V at 0 consisting of balanced open sets in V , by taking Cartesian products of balanced open subsets of the V_j 's. Similarly, if $|\cdot|$ is the trivial absolute value function on k , and if there is a local base for the topology of V_j at 0 consisting of balanced open sets for each $j \in I$, then there is a local base for the strong product topology on V at 0 consisting of balanced open sets, for the same reasons as before.

32 Direct sums

Let I be a nonempty set again, and let A_j be a commutative group for each $j \in I$. As in Section 23, the Cartesian product

$$(32.1) \quad \prod_{j \in I} A_j$$

is a commutative group with respect to coordinatewise addition, and is known as the direct product of the A_j 's. The corresponding *direct sum* is denoted

$$(32.2) \quad \sum_{j \in I} A_j,$$

and is the subgroup of the direct product consisting of elements a of (32.1) whose j th coordinate a_j is equal to 0 for all but finitely many $j \in I$. Of course, the direct sum of the A_j 's is the same as the direct product when I has only finitely many elements.

Suppose now that A_j is a commutative topological group for each $j \in I$. Note that (32.2) is dense in (32.1) with respect to the product topology on (32.1) associated to the given topology on A_j for each j . However, if $\{0\}$ is a closed set in A_j for each $j \in I$, then (32.2) is a closed subset of (32.1) with respect to the strong product topology. To see this, let a be an element of (32.1) which is not an element of (32.2), so that $a_j \neq 0$ for infinitely many $j \in I$. Put

$$(32.3) \quad U_j = A_j \setminus \{0\}$$

for every $j \in I$ such that $a_j \neq 0$, and $U_j = A_j$ for every $j \in I$ such that $a_j = 0$. Because $\{0\}$ is a closed set in A_j for each $j \in I$, by hypothesis, $A_j \setminus \{0\}$ is an open set in A_j for every $j \in I$. This implies that

$$(32.4) \quad U = \prod_{j \in I} U_j$$

is an open set in (32.1) with respect to the strong product topology. By construction, $a \in U$, and every element of U is not in the direct sum (32.2). It follows that the complement of (32.2) in (32.1) is an open set with respect to the strong product topology, so that (32.2) is a closed set with respect to the strong product topology.

As in the previous section, the direct product (32.1) is a commutative topological group with respect to the strong product topology, which implies that the direct sum (32.2) is a commutative topological group with respect to the induced topology. Let us continue to ask that $\{0\}$ be a closed set in A_j for each $j \in I$, and suppose that E is a totally bounded subset of (32.2) with respect to the topology induced on (32.2) by the strong product topology on (32.1), as in Section 27. Put

$$(32.5) \quad I(E) = \{j \in I : \text{there is an } a \in E \text{ such that } a_j \neq 0\},$$

and let us show that $I(E)$ has only finitely many elements under these conditions. Suppose for the sake of a contradiction that $I(E)$ has infinitely many elements, and for each $j \in I(E)$, let $a(j)$ be an element of E such that

$$(32.6) \quad a_j(j) \neq 0.$$

If we put

$$(32.7) \quad U_j = A_j \setminus \{a_j(j)\}$$

for each $j \in I(E)$, and $U_j = A_j$ when $j \in I \setminus I(E)$, then

$$(32.8) \quad U = \left(\sum_{j \in I} A_j \right) \cap \left(\prod_{j \in I} U_j \right)$$

is an open subset of the direct sum (32.2) with respect to the topology induced by the strong product topology on the direct product (32.1). This uses the hypothesis that $\{0\}$ be a closed set in A_j for each $j \in I$ to get that (32.7) is an open set in A_j for every $j \in I(E)$. By construction, $0 \in U_j$ for every $j \in I$, which implies that $0 \in U$.

If E is totally bounded as a subset of the direct sum (32.2) with respect to the topology induced by the strong product topology on the direct product (32.1), then there are finitely many elements $b(1), \dots, b(n)$ of (32.2) such that

$$(32.9) \quad E \subseteq \bigcup_{l=1}^n (b(l) + U).$$

Of course, for each $l = 1, \dots, n$, we have that $b_j(l) = 0$ for all but finitely many $j \in I$, because $b(l)$ is an element of (32.2). If $I(E)$ has infinitely many elements, then there is a $j_0 \in I(E)$ such that

$$(32.10) \quad b_{j_0}(l) = 0$$

for each $l = 1, \dots, n$. Let $a(j_0) \in E$ be as in the previous paragraph, so that $a_{j_0}(j_0) \notin U_{j_0}$ by the definition (32.7) of U_{j_0} . This implies that

$$(32.11) \quad a_{j_0}(j_0) \notin b_{j_0}(l) + U_{j_0}$$

for each $l = 1, \dots, n$, because of (32.10). It follows that

$$(32.12) \quad a(j_0) \notin b(l) + U$$

for each $l = 1, \dots, n$, by the definition (32.8) of U . This contradicts (32.9), since $a(j_0) \in E$. Thus $I(E)$ has only finitely many elements in this situation, as desired.

33 Direct sums, continued

Let k be a field, let I be a nonempty set, and let V_j be a vector space over k for each $j \in I$. As in Section 23, the Cartesian product

$$(33.1) \quad \prod_{j \in I} V_j$$

is a vector space over k with respect to coordinatewise addition and scalar multiplication, and is known as the direct product of the V_j 's. The *direct sum*

$$(33.2) \quad \sum_{j \in I} V_j$$

of the V_j 's is the linear subspace of the direct product (33.1) consisting of the vectors v in (33.1) whose j th coordinate v_j is equal to 0 for all but finitely many $j \in I$. Of course, a vector space over k is a commutative group with respect to addition in particular, and the commutative groups corresponding to the direct sum or product of the V_j 's as vector spaces over k are the same as the direct sum or product of the V_j 's as commutative groups, respectively. As before, the direct sum of the V_j 's is the same as the direct product when I has only finitely many elements.

Suppose from now on in this section that $|\cdot|$ is a q -absolute value function on k for some $q > 0$, and that V_j is a topological vector space with respect to $|\cdot|$ on k for each $j \in I$. Under these conditions, one can check that the direct sum (33.2) is a topological vector space with respect to $|\cdot|$ on k as well, using the topology induced on the direct sum by the strong product topology on the direct product (33.1). The main point is that if v is an element of (33.2), then $t \mapsto t v$ defines a continuous mapping from k into (33.2), with respect to the topology induced on (33.2) by the strong product topology on (33.1). As in Section 31, the analogous statement for v in the direct product (33.1) does not necessarily hold when I has infinitely many elements. However, some other properties of the direct sum can be derived from analogous statements for the direct product, as in Section 31.

Suppose now in addition that $|\cdot|$ is not the trivial absolute value function on k , and that $\{0\}$ is a closed set in V_j for each $j \in I$. Let E be a bounded subset of the direct sum (33.2), with respect to the topology on (33.2) induced by the strong product topology on the direct product (33.1), as in Section 28. As in the previous section, we put

$$(33.3) \quad I(E) = \{j \in I : \text{there is a } v \in E \text{ such that } v_j \neq 0\},$$

and we would like to show that $I(E)$ has only finitely many elements under these conditions. Suppose for the sake of a contradiction that $I(E)$ has infinitely many elements, and let $\{j_l\}_{l=1}^{\infty}$ be an infinite sequence of distinct elements of $I(E)$. Thus for each $l \in \mathbf{Z}_+$ there is a vector $v(j_l) \in E$ such that $v_{j_l}(j_l) \neq 0$. Let t_0 be an element of k such that $|t_0| > 1$, which exists because $|\cdot|$ is nontrivial on k . Observe that

$$(33.4) \quad V_{j_l} \setminus \{t_0^{-l} v_{j_l}(j_l)\}$$

is an open set in V_{j_l} that contains 0 for each $l \in \mathbf{Z}_+$, because $\{0\}$ is a closed set in V_j for each $j \in I$. It follows that there is a nonempty balanced open set $U_{j_l} \subseteq V_{j_l}$ contained in (33.4) for each $l \in \mathbf{Z}_+$, as in Section 20, using the nontriviality of $|\cdot|$ on k again. Put $U_j = V_j$ for every $j \in I$ such that $j \neq j_l$ for each $l \in \mathbf{Z}_+$, so that U_j is an open subset of V_j that contains 0 for every $j \in I$. This implies that

$$(33.5) \quad U = \left(\sum_{j \in I} V_j \right) \cap \left(\prod_{j \in I} U_j \right)$$

is an open subset of the direct sum (33.2) with respect to the topology induced by the strong product topology on the direct product (33.1), and that $0 \in U$.

If E is a bounded subset of the direct sum (33.2) with respect to this topology, then it follows that $E \subseteq tU$ for every $t \in k$ such that $|t|$ is sufficiently large. In particular, if $|t|$ is sufficiently large, then

$$(33.6) \quad v_{j_l} \in tU$$

for every $l \in \mathbf{Z}_+$, because $v(j_l) \in E$. Using the definition (33.5) of U , we get that if $|t|$ is sufficiently large, then

$$(33.7) \quad v_{j_l}(j_l) \in tU_{j_l}$$

for every $l \in \mathbf{Z}_+$. However, we also have that

$$(33.8) \quad v_{j_l}(j_l) \notin t_0^l U_{j_l}$$

for every $l \in \mathbf{Z}_+$, because U_{j_l} is contained in (33.4) for every l , by construction. This implies that

$$(33.9) \quad v_{j_l}(j_l) \notin tU_{j_l}$$

when $|t| \leq |t_0|^l$, because U_{j_l} is supposed to be balanced in V_{j_l} for every l . If $t \in k$ is given, then $|t| \leq |t_0|^l$ for all but finitely many $l \in \mathbf{Z}_+$, because $|t_0| > 1$, as in the preceding paragraph. This leads to a contradiction, so that $I(E)$ should have only finitely many elements, as desired.

34 Combining semimetrics

Let X be a set, let q be a positive real number, and let d_1, \dots, d_l be finitely many q -semimetrics on X . If $q \leq r < \infty$, then

$$(34.1) \quad \left(\sum_{j=1}^l d_j(x, y)^r \right)^{1/r}$$

is a q -semimetric on X as well. To see this, the main point is to verify that (34.1) satisfies the q -semimetric version of the triangle inequality. This is easy to do when $r = q$, and otherwise one can use Minkowski's inequality, which is the triangle inequality for the $\ell^{r/q}$ norm when $r/q \geq 1$. There is an analogous statement for infinite families of q -semimetrics on X , as long as the infinite sum corresponding to the finite sum in (34.1) is finite for every $x, y \in X$.

The analogue of (34.1) for $r = \infty$ is

$$(34.2) \quad \max_{1 \leq j \leq l} d_j(x, y).$$

It is easy to check directly that (34.2) satisfies the q -semimetric version of the triangle inequality, and hence defines a q -semimetric on X . This also works when $q = \infty$, which is to say that if d_1, \dots, d_l are semi-ultrametrics on X , then (34.2) is a semi-ultrametric on X too. Similarly, the supremum of an infinite family of q -semimetrics on X is a q -semimetric on X , as long as the supremum is finite for every $x, y \in X$. As before, this works when $q = \infty$, so that the supremum of an infinite family of semi-ultrametrics on X is a semi-ultrametric on X , as long as the supremum is finite for every $x, y \in X$.

Now let I be a nonempty set, and let X_j be a set for each $j \in I$. Also let q be a positive real number, and let \mathcal{M}_j be a nonempty collection of q -semimetrics on X_j for each $j \in I$. As in the previous paragraphs, we ask here that these q -semimetrics use the same q , which can always be arranged as in Section 7. Let us also ask that for each $j \in I$, the collection of open balls in X_j associated to elements of \mathcal{M}_j form a base for the topology on X_j determined by \mathcal{M}_j , and not just a sub-base, as in Section 2. In particular, this holds when the maximum of any finite number of elements of \mathcal{M}_j is an element of \mathcal{M}_j , which can easily be arranged by adding these maxima to \mathcal{M}_j , if necessary.

Let d_j be an element of \mathcal{M}_j for some $j \in I$, and let a_j be a positive real number, so that $a_j d_j(\cdot, \cdot)$ is a q -semimetric on X_j too. Note that

$$(34.3) \quad B_{a_j d_j}(x_j, r) = B_{d_j}(x_j, r/a_j)$$

for every $x_j \in X_j$ and $r > 0$, where these open balls in X_j centered at x_j associated to d_j and $a_j d_j$ are defined as in (1.5), as usual. Put

$$(34.4) \quad d'_j(x_j, y_j) = \min(a_j d_j(x_j, y_j), 1)$$

for every $x_j, y_j \in X_j$, which defines a q -semimetric on X_j as well, as in Section 24. If $0 < r \leq 1$, then

$$(34.5) \quad B_{d'_j}(x_j, r) = B_{a_j d_j}(x_j, r) = B_{d_j}(x_j, r/a_j)$$

for every $x_j \in X_j$, where $B_{d'_j}(x_j, r)$ is the open ball in X_j centered at x_j with radius r associated to d'_j . The first equality in (34.5) corresponds to (24.2), and we have that

$$(34.6) \quad B_{d'_j}(x_j, r) = X_j$$

for every $x_j \in X_j$ when $r > 1$, as in (24.3).

As in Section 23, let

$$(34.7) \quad X = \prod_{j \in I} X_j$$

be the Cartesian product of the X_j 's, and let $x_j \in X_j$ be the j th component of $x \in X$ for each $j \in I$. Also let X be equipped with the strong product topology corresponding to the topology on X_j determined by \mathcal{M}_j for each $j \in I$. Suppose that $d_j \in \mathcal{M}_j$ and $a_j > 0$ are given as in the preceding paragraph for each $j \in I$, and put

$$(34.8) \quad d(x, y) = \sup_{j \in I} d'_j(x_j, y_j)$$

for every $x, y \in X$, where d'_j is associated to d_j and a_j as in (34.4). Of course,

$$(34.9) \quad d(x, y) \leq 1$$

for every $x, y \in X$, because $d'_j(x_j, y_j) \leq 1$ for each $j \in I$, by construction. Thus (34.8) is finite, and hence defines a q -semimetric on X under these conditions, as mentioned earlier.

Observe that

$$(34.10) \quad B_d(x, r) \subseteq \prod_{j \in I} B_{d'_j}(x_j, r)$$

for every $x \in X$ and $r > 0$, where $B_d(x, r)$ is the open ball in X centered at x with radius r associated to d as in (1.5), and $B_{d'_j}(x_j, r)$ is the open ball in X_j centered at x_j with radius r associated to d'_j for each $j \in I$, as before. More precisely, one can check that

$$(34.11) \quad B_d(x, r) = \bigcup_{0 < \tilde{r} < r} \prod_{j \in I} B_{d'_j}(x_j, \tilde{r})$$

for every $x \in X$ and $r > 0$. This implies that $B_d(x, r)$ is an open set in X with respect to the strong product topology for every $x \in X$ and $r > 0$. This uses the fact that $B_{d'_j}(x_j, \tilde{r})$ is an open set in X_j for each $j \in I$, by (34.5) and (34.6), so that

$$(34.12) \quad \prod_{j \in I} B_{d'_j}(x_j, \tilde{r})$$

is an open set in X with respect to the strong product topology. One can also verify that the topology on X determined by the collection of q -semimetrics on X of the form (34.8) with $d_j \in \mathcal{M}_j$ and $a_j > 0$ is the same as the strong product topology, using (34.10). It suffices for this to use any collection of positive real numbers a_j that can be arbitrarily large, instead of all $a_j > 0$. It is here that

we use the hypothesis that the open balls in X_j associated to elements of \mathcal{M}_j form a base for the topology of X_j for each $j \in I$.

Suppose that X_j is a commutative group for each $j \in I$, so that X is also a commutative group with respect to coordinatewise addition. If d_j is a translation-invariant q -semimetric on X_j for some $j \in I$, then (34.4) is invariant under translations on X_j for every $a_j > 0$. If d_j is a translation-invariant q -semimetric on X_j and $a_j > 0$ for each $j \in I$, then it follows that (34.8) is invariant under translations on X . If every element of \mathcal{M}_j is invariant under translations on X_j for each $j \in I$, then we get a collection of translation-invariant q -semimetrics on X for which the corresponding topology is the same as the strong product topology, as in the previous paragraph.

35 Combining seminorms

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . If N_1, \dots, N_l are finitely many q -seminorms on V , and $q \leq r < \infty$, then

$$(35.1) \quad \left(\sum_{j=1}^l N_j(v)^r \right)^{1/r}$$

is a q -seminorm on V too. As in the previous section, the main point is to check that (35.1) satisfies the q -seminorm version of the triangle inequality, which is essentially the same as for the corresponding q -semimetrics. As before, the analogue of (35.1) for $r = \infty$ is

$$(35.2) \quad \max_{1 \leq j \leq l} N_j(v).$$

Similarly, if the N_j 's are semi-ultranorms on V , then (35.2) is a semi-ultranorm on V as well.

As in the previous section again, there are analogous statements for infinite families of q -seminorms, as long as the relevant quantities are finite. In particular, this can be applied to direct sums of vector spaces. Let I be a nonempty set, let V_j be a vector space over k for each $j \in I$, and let

$$(35.3) \quad V = \sum_{j \in I} V_j$$

be the corresponding direct sum, as in Section 33. If N_j is a q -seminorm on V_j for each $j \in I$, and if $q \leq r < \infty$, then

$$(35.4) \quad \left(\sum_{j \in I} N_j(v_j)^r \right)^{1/r}$$

defines a q -seminorm on V . Here $v_j \in V_j$ is the j th component of $v \in V$ for each $j \in I$, as usual, which is equal to 0 for all but finitely many $j \in I$, by the

definition of the direct sum. This implies that $N_j(v_j) = 0$ for all but finitely many $j \in I$, so that the sum in (35.4) is finite. Similarly, the maximum

$$(35.5) \quad \max_{j \in I} N_j(v_j)$$

is attained for every $v \in V$, and defines a q -seminorm on V too. If N_j is a semi-ultranorm on V_j for each $j \in I$, then (35.5) defines a semi-ultranorm on V .

Suppose now that \mathcal{N}_j is a nonempty collection of q -seminorms on V_j for each $j \in I$, where more precisely the same q should be used for every element of \mathcal{N}_j for each $j \in I$. Let V_j be equipped with the topology determined by the collection of q -semimetrics associated to the elements of \mathcal{N}_j for each $j \in I$, and let V be equipped with the topology induced by the corresponding strong product topology on the Cartesian product of the V_j 's. As in the preceding section, we ask that the open balls in V_j centered at 0 associated to elements of \mathcal{N}_j form a local base for the topology of V_j at 0 for each $j \in I$, and not just a sub-base. As before, this holds automatically when the maximum of any finite number of elements of \mathcal{N}_j is an element of \mathcal{N}_j for each $j \in I$, which can always be arranged by adding these maxima to \mathcal{N}_j .

Let N_j be an element of \mathcal{N}_j for each $j \in I$, and let a_j be a positive real number for each $j \in I$. Thus $a_j N_j$ is a q -seminorm on V_j for each $j \in I$, so that

$$(35.6) \quad N(v) = \max_{j \in I} (a_j N_j(v_j))$$

defines a q -seminorm on V , as in (35.5). As in (34.3),

$$(35.7) \quad B_{a_j N_j}(0, r) = B_{N_j}(0, r/a_j)$$

for every $j \in I$ and $r > 0$, where these open balls in V_j associated to N_j and $a_j N_j$ are defined as in (10.7). It is easy to see that

$$(35.8) \quad B_N(0, r) = V \cap \left(\prod_{j \in I} B_{a_j N_j}(0, r) \right)$$

for every $r > 0$, where the left side of (35.8) is the open ball in V associated to N as in (10.7). Remember that the direct sum V is contained in the Cartesian product of the V_j 's, which also contains the product of open balls in the V_j 's on the right side of (35.8). This step is a bit simpler than its analogue in the previous section, because N is defined in (35.6) as a maximum, instead of a supremum. It follows from (35.7) and (35.8) that

$$(35.9) \quad B_N(0, r) = V \cap \left(\prod_{j \in I} B_{N_j}(0, r/a_j) \right)$$

for every $r > 0$. Of course,

$$(35.10) \quad \prod_{j \in I} B_{N_j}(0, r/a_j)$$

is an open set in the Cartesian product of the V_j 's with respect to the strong product topology for every $r > 0$, since open balls in V_j with respect to N_j are open subsets of V_j for each $j \in I$ by construction. This implies that (35.9) is an open set in V with respect to the topology induced by the strong product topology on the Cartesian product of the V_j 's. Using (35.9), one can also check that the topology induced on V by the strong product topology on the Cartesian product of the V_j 's is the same as the topology determined by the collection of q -seminorms of the form (35.6), where $N_j \in \mathcal{N}_j$ and $a_j > 0$ for each $j \in I$. As in the previous section, it suffices to use any collection of positive real numbers a_j that can be arbitrarily large here, instead of all $a_j > 0$. It is also here that we use the hypothesis that open balls in V_j centered at 0 associated to elements of \mathcal{N}_j form a local base for the topology of V_j at 0 for each $j \in I$, as before.

36 Continuous linear mappings, continued

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V, W be vector spaces over k . Also let $\mathcal{N}_V, \mathcal{N}_W$ be nonempty collections of q -seminorms on V, W , respectively. As before, one can let $q > 0$ depend on the elements of \mathcal{N}_V and \mathcal{N}_W , as long as $|\cdot|$ is a q -absolute value function on k for that choice of q . Note that for any finite collection of elements of \mathcal{N}_V or \mathcal{N}_W , there is a $q > 0$ that works for each element of the finite collection, by taking the minimum of the corresponding finitely many q 's. Let V, W be equipped with the topologies determined by $\mathcal{N}_V, \mathcal{N}_W$, respectively, so that V and W are topological vector spaces.

Let ϕ be a linear mapping from V into W . It is easy to see that ϕ is continuous at 0 if and only if for every $N_W \in \mathcal{N}_W$ and $r > 0$ there are finitely many elements $N_{V,1}, \dots, N_{V,l}$ of \mathcal{N}_V and finitely many positive real numbers r_1, \dots, r_l such that

$$(36.1) \quad \phi\left(\bigcap_{j=1}^l B_{N_{V,j}}(0, r_j)\right) \subseteq B_{N_W}(0, r).$$

Here $B_{N_W}(0, r)$ is the open ball in W associated to N_W and r as in (10.7), and similarly $B_{N_{V,j}}(0, r_j)$ is the open ball in V associated to $N_{V,j}$ and r_j for $j = 1, \dots, l$. In this case, we get that

$$(36.2) \quad \phi\left(\bigcap_{j=1}^l B_{N_{V,j}}(0, |t|r_j)\right) \subseteq B_{N_W}(0, |t|r)$$

for every $t \in k$ with $t \neq 0$, using (10.9). If $|\cdot|$ is nontrivial on k , then it follows that a condition like (36.1) for a single $r > 0$ implies an analogous condition for every $r > 0$.

Suppose that there is a positive real number C such that

$$(36.3) \quad N_W(\phi(v)) \leq C \max_{1 \leq j \leq l} N_{V,j}(v)$$

for every $v \in V$. This implies that for each $r > 0$, (36.1) holds with $r_j = r/C$ for $j = 1, \dots, l$. As a partial converse, if (36.1) holds for some positive real numbers r, r_1, \dots, r_l , and if $|\cdot|$ is not trivial on k , then one can check that there is a $C > 0$ such that (36.3) holds for every $v \in V$. This is a bit simpler when $|\cdot|$ is not discrete on k , so that (12.4) is dense in \mathbf{R}_+ with respect to the standard Euclidean topology on \mathbf{R} . Otherwise, if $|\cdot|$ is nontrivial and discrete on k , then the constant C just mentioned also depends on the quantity (12.5) associated to $|\cdot|$ on k .

Suppose now that $|\cdot|$ is the trivial absolute value function on k , and that \mathcal{N}_V consists of only the trivial ultranorm on V . This implies that the topology on V determined by \mathcal{N}_V is the same as the discrete topology, so that any mapping from V into any topological space is continuous. In this situation, (36.3) would say that

$$(36.4) \quad N_W(\phi(v)) \leq C$$

for every $v \in V$ with $v \neq 0$. If V is finite-dimensional over k , then there is always a $C > 0$ so that this holds. Otherwise, if V is infinite-dimensional over k , then one can give examples where this does not work, with $W = V$ and ϕ equal to the identity mapping.

37 Direct sums, revisited

Let I be a nonempty set, let A_j be a commutative group for each $j \in I$, and let

$$(37.1) \quad A = \sum_{j \in I} A_j$$

be the corresponding direct sum, as in Section 32. Note that for each $l \in I$, there is a natural embedding of A_l into A , which sends each element of A_l to the element of A whose l th coordinate is that element of A_l , and whose j th coordinate is equal to 0 for every $j \in I$ with $j \neq l$. If ϕ is a homomorphism from A into another commutative group B , then one gets a homomorphism ϕ_l from A_l into B for each $l \in I$, by composing the natural embedding of A_l into A with ϕ . In the other direction, if ϕ_j is a homomorphism from A_j into B for each $j \in I$, then

$$(37.2) \quad \phi(a) = \sum_{j \in I} \phi_j(a_j)$$

defines a homomorphism from A into B . Remember that for each $a \in A$, $a_j = 0$ for all but finitely $j \in I$, which implies that $\phi_j(a_j) = 0$ for all but finitely many $j \in I$, so that the sum in (37.2) makes sense. If the A_j 's are vector spaces over a field k , then A is also a vector space over k , and the natural embedding of A_l into A is linear for each $l \in I$. If B is another vector space over k , then linear mappings from A into B correspond to families of linear mappings from the A_j 's into B as before.

Suppose now that A_j is a commutative topological group for each $j \in I$, and let A be equipped with the topology induced by the corresponding strong

product topology on the Cartesian product of the A_j 's, as in Section 32. It is easy to see that for each $l \in I$, the natural embedding from A_l into A is a homeomorphism onto its image with respect to the induced topology. Let B be another commutative topological group, and suppose that ϕ is a continuous homomorphism from A into B . If ϕ_j is the corresponding homomorphism from A_j into B for each $j \in I$, as in the preceding paragraph, then ϕ_j is also continuous for each j , since it is the composition of two continuous mappings.

Now let ϕ_j be a continuous homomorphism from A_j into B for each $j \in I$, and let ϕ be the corresponding homomorphism from A into B , as in (37.2). If I has only finitely many elements, then one can check that ϕ is also continuous. More precisely, put

$$(37.3) \quad \tilde{\phi}_j(a) = \phi_j(a_j)$$

for every $a \in A$ and $j \in I$, which is the same as the composition of ϕ_j with the natural coordinate projection from A on A_j . Thus $\tilde{\phi}_j$ is continuous as a mapping from A into B for each $j \in I$, because it is the composition of two continuous mappings. If I has only finitely many elements, then it follows that ϕ is continuous as a mapping from A into B as well, since it is the sum of finitely many continuous mappings from A into B .

In order to prove an analogous statement when I is countably infinite, let us begin with some consequences of continuity of addition on B . Let W be an open subset of B that contains 0, and let $W_1 \subseteq B$ be an open set such that $0 \in W_1$ and

$$(37.4) \quad W_1 + W_1 \subseteq W.$$

Continuing in this way, we get for each integer $n \geq 2$ an open set $W_n \subseteq B$ such that $0 \in W_n$ and

$$(37.5) \quad W_n + W_n \subseteq W_{n-1}.$$

This implies that

$$(37.6) \quad W_1 + W_2 + \cdots + W_{n-1} + W_n + W_n \subseteq W_1 + W_2 + \cdots + W_{n-1} + W_{n-1}$$

when $n \geq 2$, and hence that

$$(37.7) \quad W_1 + \cdots + W_n + W_n \subseteq W$$

for every $n \geq 1$. In particular,

$$(37.8) \quad W_1 + \cdots + W_n \subseteq W$$

for each n , since $0 \in W_n$.

To deal with the case where I is countably infinite, we may as well suppose that $I = \mathbf{Z}_+$. Let W be given as in the previous paragraph, and let W_n be chosen for $n \in \mathbf{Z}_+$ as before. Suppose that ϕ_j is a continuous homomorphism from A_j into B for each $j \in I = \mathbf{Z}_+$, and let U_j be an open subset of A_j such that $0 \in U_j$ and

$$(37.9) \quad \phi_j(U_j) \subseteq W_j$$

for each j . Put

$$(37.10) \quad U = A \cap \left(\prod_{j=1}^{\infty} U_j \right),$$

which is an open subset of A with respect to the topology induced by the strong product topology on $\prod_{j=1}^{\infty} A_j$. Note that $0 \in U$, since $0 \in U_j$ for each j . One can also check that

$$(37.11) \quad \phi(U) \subseteq W,$$

where ϕ is as in (37.3), using (37.8) and (37.9). This implies that ϕ is continuous at 0 with respect to the topology on A induced by the strong product topology on $\prod_{j=1}^{\infty} A_j$, and hence that ϕ is continuous everywhere on A with respect to this topology, because ϕ is a homomorphism.

Here is another version of this type of argument. Suppose that $d_B(\cdot, \cdot)$ is a q -semimetric on B for some $q > 0$ that is invariant under translations, and compatible with the given topology on B . Let $\epsilon > 0$ be given. If I has only finitely or countably many elements, then one can choose $\epsilon_j > 0$ for every $j \in I$ so that

$$(37.12) \quad \sum_{j \in I} \epsilon_j^q \leq \epsilon^q,$$

which is the same as saying that the sum of ϵ_j^q over any finite subset of I is less than or equal to ϵ^q . Suppose that ϕ_j is a continuous homomorphism from A_j into B for each $j \in I$, so that ϕ_j is continuous at 0 with respect to $d_B(\cdot, \cdot)$ on B in particular. This implies that for each $j \in I$, there is an open set $U_j \subseteq A_j$ such that $0 \in A_j$ and

$$(37.13) \quad \phi_j(U_j) \subseteq B_{d_B}(0, \epsilon_j),$$

where the right side of (37.13) is the open ball in B with respect to $d_B(\cdot, \cdot)$ centered at 0 with radius ϵ_j , as in (1.5). As before,

$$(37.14) \quad U = A \cap \left(\prod_{j \in I} U_j \right)$$

is an open subset of A with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$, and $0 \in U$. Using (37.12), (37.13), and the q -semimetric version of the triangle inequality, one can verify that

$$(37.15) \quad \phi(U) \subseteq B_{d_B}(0, \epsilon),$$

where the right side of (37.15) is the open ball in B with respect to $d_B(\cdot, \cdot)$ centered at 0 with radius ϵ . This means that ϕ is continuous at 0 with respect to $d_B(\cdot, \cdot)$ on B and the topology on A induced by the strong product topology on $\prod_{j \in I} A_j$.

Suppose now that $d_B(\cdot, \cdot)$ is a semi-ultrametric on B that is invariant under translations and compatible with the given topology on B , which corresponds to $q = \infty$ in the preceding paragraph. In this case, the analogous argument works for any nonempty set I , with $\epsilon_j = \epsilon$ for every $j \in I$.

38 Combining seminorms, continued

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let I be a nonempty set. Also let V_j be a vector space over k for each $j \in I$, and let

$$(38.1) \quad V = \sum_{j \in I} V_j$$

be the corresponding direct sum, as in Section 33. If N_j is a q -seminorm on V_j for each $j \in I$, then

$$(38.2) \quad \left(\sum_{j \in I} N_j(v_j)^r \right)^{1/r}$$

defines a q -seminorm on V when $q \leq r < \infty$, as in Section 35. Similarly,

$$(38.3) \quad \max_{j \in I} N_j(v_j)$$

defines a q -seminorm on V too, as before. If I has only finitely many elements, then (38.2) is bounded by (38.3) times the r th root of the number of elements of I .

Suppose now that I is countably infinite, and let a_j be a positive real number for each $j \in I$ such that

$$(38.4) \quad \sum_{j \in I} a_j^{-r} < \infty,$$

where the sum is defined as the supremum of the corresponding finite subsums. Observe that

$$(38.5) \quad \begin{aligned} \left(\sum_{j \in I} N_j(v_j)^r \right)^{1/r} &= \left(\sum_{j \in I} a_j^{-r} (a_j N_j(v_j))^r \right)^{1/r} \\ &\leq \left(\sum_{j \in I} a_j^{-r} \right)^{1/r} \max_{j \in I} (a_j N_j(v_j)) \end{aligned}$$

for every $v \in V$. Of course,

$$(38.6) \quad \max_{j \in I} (a_j N_j(v_j))$$

also defines a q -seminorm on V , as in (38.3).

Let I be any nonempty set again, and for each $l \in I$, let η_l be the obvious inclusion mapping from V_l into V . Thus for each $l \in I$ and $v_l \in V_l$, $\eta_l(v_l)$ is the element of V whose l th coordinate is equal to v_l , and whose j th coordinate is equal to 0 for every $j \in I$ with $j \neq l$. Let N be a q -seminorm on V , and suppose that for each $l \in I$ there is a nonnegative real number C_l such that

$$(38.7) \quad N(\eta_l(v_l)) \leq C_l N_l(v_l)$$

for every $v_l \in V_l$. Using the q -semimetric version of the triangle inequality, we get that

$$(38.8) \quad N(v) \leq \left(\sum_{j \in I} C_l^q N_l(v_l)^q \right)^{1/q}$$

for every $v \in V$. Note that the right side of (38.8) is a q -semimetric on V , as in (38.2).

In the analogous situation for $q = \infty$, we ask that N_l be a semi-ultranorm on V_l for each $l \in I$, and that N be a semi-ultranorm on V . In this case, (38.7) implies that

$$(38.9) \quad N(v) \leq \max_{j \in I} (C_l N_l(v_l))$$

for every $v \in V$. As before, the right side of (38.9) is a semi-ultranorm on V too.

39 Completeness

Let X be a set, and let $d(x, y)$ be a q -semimetric on X for some $q > 0$. As usual, a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of X is said to be a *Cauchy sequence* with respect to $d(\cdot, \cdot)$ if

$$(39.1) \quad \lim_{j, l \rightarrow \infty} d(x_j, x_l) = 0.$$

Of course, this happens if and only if

$$(39.2) \quad \lim_{j, l \rightarrow \infty} d(x_j, x_l)^q = 0,$$

so that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to $d(\cdot, \cdot)^q$ as an ordinary semimetric on X . If $\{x_j\}_{j=1}^{\infty}$ converges to an element of X with respect to the topology determined by $d(\cdot, \cdot)$, then it is easy to see that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to $d(\cdot, \cdot)$, using the q -semimetric version of the triangle inequality. Conversely, if every Cauchy sequence of elements of X with respect to $d(\cdot, \cdot)$ converges to an element of X with respect to the topology determined by $d(\cdot, \cdot)$, then X is said to be *complete* with respect to $d(\cdot, \cdot)$.

Now let A be a commutative topological group. A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of A is said to be a *Cauchy sequence* in A if

$$(39.3) \quad \lim_{j, l \rightarrow \infty} (x_j - x_l) = 0.$$

Equivalently, this means that for each open set $U \subseteq A$ that contains 0, there is a positive integer L such that

$$(39.4) \quad x_j - x_l \in U$$

for every $j, l \geq 1$. As before, one can check that if $\{x_j\}_{j=1}^{\infty}$ converges to an element of A , then $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in A . This uses the continuity of addition on A at 0.

Let $d(x, y)$ be a translation-invariant q -semimetric on A for some $q > 0$. Suppose that $d(x, y)$ is compatible with the given topology on A , in the sense that $d(x, 0)$ is continuous as a real-valued function on A at $x = 0$ with respect to the standard topology on \mathbf{R} , as in Sections 17 and 19. If $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in A as a commutative topological group, as in the preceding paragraph,

then $\{x_j\}_{j=1}^\infty$ is also a Cauchy sequence with respect to $d(\cdot, \cdot)$. Suppose for the moment that the topology on A is determined by a nonempty collection \mathcal{M} of translation-invariant q -semimetrics on A , where $q > 0$ is allowed to depend on the element of \mathcal{M} . In this case, a sequence $\{x_j\}_{j=1}^\infty$ of elements of A is a Cauchy sequence in A as a commutative topological group if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to each element of \mathcal{M} .

If every Cauchy sequence of elements of A as a commutative topological group converges to an element of A , then A is said to be *sequentially complete*. If the topology on A is determined by a single translation-invariant q -semimetric $d(x, y)$ for some $q > 0$, then A is sequentially complete as a commutative topological group if and only if A is complete with respect to $d(x, y)$. Remember that if there is a local base for the topology of A at 0 with only finitely or countably many elements, then there is a translation-invariant semimetric on A that determines the same topology on A , as in Section 25. In this case, it is reasonable to simply say that A is complete as a commutative topological group when A is sequentially complete. Otherwise, one would normally define completeness of A in terms of convergence of Cauchy nets in A , or Cauchy filters on A .

40 Equicontinuity

Let A_1, A_2 be commutative topological groups, and let \mathcal{L} be a collection of group homomorphisms from A_1 into A_2 . We say that \mathcal{L} is *equicontinuous* at 0 if for each open set $U_2 \subseteq A_2$ that contains 0 there is an open set $U_1 \subseteq A_1$ such that $0 \in U_1$ and

$$(40.1) \quad \phi(U_1) \subseteq U_2$$

for every $\phi \in \mathcal{L}$. Of course, this implies that each element of \mathcal{L} is continuous at 0. If \mathcal{L} has only finitely many elements, and each element of \mathcal{L} is continuous at 0, then \mathcal{L} is equicontinuous at 0. If A_1 is equipped with the discrete topology, then the collection of all group homomorphisms from A_1 into A_2 is equicontinuous at 0.

Now let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a topological vector space over k . Put

$$(40.2) \quad \delta_a(v) = a v$$

for each $a \in k$ and $v \in V$, so that δ_a is a linear mapping from V into itself for each $a \in k$. If $|\cdot|$ is not the trivial absolute value function on k , then

$$(40.3) \quad \{\delta_a : a \in k, |a| \leq 1\}$$

is equicontinuous at 0 on V . This uses the fact that balanced nonempty open subsets of V form a local base for the topology of V at 0 when $|\cdot|$ is nontrivial on k , as in Section 20. Under these conditions, it follows that

$$(40.4) \quad \{\delta_a : a \in k, |a| \leq |b|\}$$

is equicontinuous at 0 for each $b \in k$. This is the same as saying that

$$(40.5) \quad \{\delta_a : a \in k, |a| \leq r\}$$

is equicontinuous at 0 on V for every positive real number r . More precisely, if $|\cdot|$ is nontrivial on k , then there is a $b_0 \in k$ such that $|b_0| > 1$. Thus for each $r > 0$ there is an $n \in \mathbf{Z}_+$ such that

$$(40.6) \quad r \leq |b_0|^n = |b_0^n|,$$

so that (40.5) is contained in (40.4) with $b = b_0^n$.

If $|\cdot|$ is the trivial absolute value function on k , then the equicontinuity of

$$(40.7) \quad \{\delta_a : a \in k\}$$

at 0 corresponds to (20.6). As in Section 20, this is equivalent to the condition that nonempty balanced open subsets of V form a local base for the topology of V at 0, which is not automatic in this case.

Suppose again that $|\cdot|$ is nontrivial on k , and let V_1, V_2 be topological vector spaces over k . Thus the notion of bounded subsets of V_1, V_2 can be defined as in Section 28. Let \mathcal{L} be a collection of linear mappings from V_1 into V_2 . If \mathcal{L} is equicontinuous at 0, and if E_1 is a bounded subset of V_1 , then one can check that

$$(40.8) \quad \bigcup_{\phi \in \mathcal{L}} \phi(E_1)$$

is a bounded subset of V_2 . If E_1 is an open subset of V_1 that contains 0, and if (40.8) is a bounded subset of V_2 , then it is easy to see that \mathcal{L} is equicontinuous at 0.

Part II

Topological dimension 0

41 Separated sets

As usual, a pair A, B of subsets of a topological space X are said to be *separated* in X if

$$(41.1) \quad \overline{A} \cap B = A \cap \overline{B} = \emptyset,$$

where $\overline{A}, \overline{B}$ are the closures of A, B in X , respectively. In particular, disjoint closed subsets of X are separated, and it is easy to see that disjoint open sets in X are separated too. If $A, B \subseteq X$ are separated and $A \cup B = X$, then both A and B are both open and closed in X . If $Y \subseteq X$, then Y may also be considered as a topological space, with respect to the topology induced by the one on X . It is well known and easy to check that $A, B \subseteq Y$ are separated with respect to the induced topology on Y if and only if A, B are separated as subsets of X . A

set $E \subseteq X$ is said to be *connected* if it cannot be expressed as the union of two nonempty separated sets. Thus X is connected if it cannot be expressed as the union of two nonempty disjoint closed sets, or equivalently as the union of two nonempty disjoint open sets. If $E \subseteq Y \subseteq X$, then E is connected with respect to the induced topology on Y if and only if E is connected as a subset of X . One can check that the closure of a connected subset of X is also connected in X . A subset of X is said to be *totally disconnected* if it does not contain any connected set with at least two elements.

Now let A be a commutative topological group, and let $U \subseteq A$ be an open set that contains 0. It will be convenient to ask that U also be symmetric about 0 in A , in the sense that

$$(41.2) \quad -U = U,$$

which can always be arranged by replacing U with $U \cap (-U)$. Let us say that $B, C \subseteq A$ are U -separated in A if

$$(41.3) \quad (B + U) \cap C = \emptyset,$$

which is equivalent to asking that

$$(41.4) \quad B \cap (C + U) = \emptyset,$$

because of (41.2). Using the continuity of the group operations on A at 0, it is easy to see that there is an open set $U_1 \subseteq A$ that contains 0, is symmetric about 0, and satisfies

$$(41.5) \quad U_1 + U_1 \subseteq U,$$

as in (18.11). Combining this with (41.3) or (41.4), one can check that

$$(41.6) \quad (B + U_1) \cap (C + U_1) = \emptyset.$$

Remember that the closures of B, C in A are contained in $B + U_1, C + U_1$, respectively, as in (18.8). Thus (41.6) implies in particular that the closures of B and C in A are disjoint.

Suppose that $C \subseteq A$ is compact, $W \subseteq A$ is an open set, and that $C \subseteq W$. If $x \in C$, then there is an open set $U(x) \subseteq A$ that contains 0 and satisfies

$$(41.7) \quad x + U(x) + U(x) \subseteq W,$$

by continuity of addition on A . The collection of open sets $x + U(x)$ with $x \in C$ covers C , and so there are finitely many elements x_1, \dots, x_n of C such that

$$(41.8) \quad C \subseteq \bigcup_{j=1}^n (x_j + U(x_j)),$$

by compactness. Put

$$(41.9) \quad U = \bigcap_{j=1}^n (U(x_j) \cap (-U(x_j))),$$

which is an open subset of A that contains 0 and is symmetric about 0. Using (41.7) and (41.8), we get that

$$(41.10) \quad C + U \subseteq \bigcup_{j=1}^n (x_j + U(x_j) + U) \subseteq \bigcup_{j=1}^n (x_j + U(x_j) + U(x_j)) \subseteq W.$$

If $E \subseteq A$ is a closed set that is disjoint from C , then we can apply the previous argument to $W = A \setminus E$. In this case, (41.10) is equivalent to saying that

$$(41.11) \quad (C + U) \cap E = \emptyset,$$

so that C and E are U -separated in A .

Let X be a topological space that is regular in the strict sense, without including the first or 0th separation condition. If $K \subseteq X$ is compact, $W \subseteq X$ is an open set, and $K \subseteq W$, then there is an open set $U \subseteq X$ such that $K \subseteq U$ and $\overline{U} \subseteq W$. More precisely, for each $x \in K$, there is an open set $U(x) \subseteq X$ such that $x \in U(x)$ and $\overline{U(x)} \subseteq W$, because X is regular. Because K is compact, K can be covered by finitely many $U(x)$'s, whose union U has the desired properties. In particular, if $K \subseteq X$ is compact and open, then one can apply the previous argument with $K = W$, to conclude that K is a closed set in X . Of course, if X is Hausdorff, then every compact subset of X is closed. However, if X is regular in the strict sense, then X need not be Hausdorff, and arbitrary compact subsets of X need not be closed. If X is any set equipped with the indiscrete topology, for instance, then X is regular in the strict sense, and every subset of X is compact.

42 Open subgroups

Let A be a commutative topological group again, and let B be a subgroup of A which is also an open set in A . It is well known that the complement of B in A can be expressed as a union of cosets of B in A , which are translates of B in A . Each translate of B is also an open set in A , and hence any union of translates of B in A is an open set in A too. This implies that the complement of B in A is an open set in A , which means that B is a closed set in A as well. In particular, if A is connected as a topological space, then A has no proper open subgroups.

If B is any subgroup of A , then

$$(42.1) \quad 0 \in B,$$

$$(42.2) \quad -B = B,$$

and

$$(42.3) \quad B + B = B.$$

Conversely, if B is any subset of A that satisfies (42.1), (42.2), and (42.3), then B is a subgroup of A . If B is an open subgroup of A , then (42.3) implies that B is B -separated from its complement in A , as discussed in the previous section.

Suppose that $U \subseteq A$ contains 0 and is symmetric about 0. Put $U_1 = U$, and define U_j recursively for $j \in \mathbf{Z}_+$ by putting

$$(42.4) \quad U_{j+1} = U_j + U$$

for each $j \geq 1$. Equivalently, U_j is the subset of A consisting of sums of exactly j elements of U for each j . Note that $0 \in U_j$, U_j is symmetric about 0, and $U_j \subseteq U_{j+1}$ for each j , and that

$$(42.5) \quad U_{j+l} = U_j + U_l$$

for each $j, l \geq 1$. If we put

$$(42.6) \quad B = \bigcup_{j=1}^{\infty} U_j,$$

then B satisfies (42.1), (42.2), and (42.3). Thus B is a subgroup of A . If U is an open set in A , then U_j is an open set in A for each j , and hence B is an open set in A too.

Let $U \subseteq A$ be an open set that contains 0 and is symmetric about 0, and suppose that E is a nonempty subset of A that is U -separated from its complement in A . This means that

$$(42.7) \quad (E + U) \cap (A \setminus E) = \emptyset,$$

as in the previous section, which is the same as saying that

$$(42.8) \quad E + U \subseteq E.$$

If U_j is defined as in the preceding paragraph for each $j \in \mathbf{Z}_+$, then it follows that

$$(42.9) \quad E + U_j \subseteq E$$

for each $j \geq 1$. This implies that

$$(42.10) \quad E + B \subseteq E,$$

where B is as in (42.6), by taking the union over $j \in \mathbf{Z}_+$ in (42.9). Of course, we actually have equality in (42.8), (42.9), and (42.10), because 0 is an element of U , and hence of U_j for each j , as well as B .

43 Semimetrics and partitions

Let X be a set, and let \mathcal{P} be a partition of X . Thus \mathcal{P} is a collection of pairwise-disjoint nonempty subsets of X whose union is equal to X . Define

$$(43.1) \quad d_{\mathcal{P}}(x, y)$$

for $x, y \in X$ by putting (43.1) equal to 0 when x and y are contained in the same element of \mathcal{P} , and equal to 1 otherwise. It is easy to see that this defines

a semi-ultrametric on X , which one might describe as the *discrete semimetric* associated to \mathcal{P} . If \mathcal{P} is the partition of X consisting of all subsets of X with exactly one element, then (43.1) is the same as the discrete metric on X . If \mathcal{P} consists of only X itself, then let us call \mathcal{P} the *trivial partition* of X . In this case, (43.1) is equal to 0 for every $x, y \in X$.

Let Z be another set, and let ϕ be a mapping from X into Z . If $d_Z(\cdot, \cdot)$ is a q -semimetric on Z for some $q > 0$, then

$$(43.2) \quad d_Z(\phi(x), \phi(y))$$

defines a q -semimetric on X . Note that

$$(43.3) \quad \{\phi^{-1}(z) : z \in \phi(X)\}$$

is a partition of X under these conditions. If $d_Z(\cdot, \cdot)$ is the discrete metric on Z , then (43.2) is the same as the discrete semimetric associated to the partition (43.3), as in the preceding paragraph. Of course, if \mathcal{P} is any partition of X , then \mathcal{P} is of the form (43.3), where $Z = \mathcal{P}$ and ϕ is the mapping that sends $x \in X$ to the element of \mathcal{P} that contains x .

If $d(x, y)$ is a q -semimetric on X for any $q > 0$, then

$$(43.4) \quad d(x, y) = 0$$

defines an equivalence relation on X , whose equivalence classes determine a partition of X . These equivalence classes are the same as the closed balls in X of radius 0 with respect to $d(\cdot, \cdot)$. Similarly, if $d(x, y)$ is a semi-ultrametric on X , then for each $r > 0$, X is partitioned by the open balls in X with respect to $d(\cdot, \cdot)$ of radius r , and by the closed balls in X of radius r . In the case of (43.1), the elements of \mathcal{P} are the same as the open balls of radius $0 < r \leq 1$, and the closed balls of radius $0 \leq r < 1$. If $d(x, y)$ is a q -semimetric on X for some $q > 0$ that takes values in the set $\{0, 1\}$, then $d(x, y)$ is the same as the discrete semimetric associated to the partition of X consisting of the closed balls of radius 0.

Let X be a topological space, and let \mathcal{P} be a partition of X . If every element of \mathcal{P} is an open subset of X , then every element of \mathcal{P} is a closed set in X too, because the complement of every element of \mathcal{P} in X is equal to the union of the other elements of \mathcal{P} , and hence is an open subset of X . In particular, if X is connected, then \mathcal{P} is the trivial partition of X under these conditions. Note that every element of \mathcal{P} is an open subset of X exactly when the corresponding discrete semimetric (43.1) is compatible with the topology on X , in the sense that open subsets of X with respect to (43.1) are also open with respect to the given topology on X . If Z is any set equipped with the discrete topology, then a mapping ϕ from X into Z is continuous if and only if (43.3) is an open set in X for every $z \in Z$.

44 Dimension 0

A topological space X is said to have *topological dimension 0* at a point $x \in X$ if for each open set $W \subseteq X$ that contains x there is an open set $W_1 \subseteq W$ such that $x \in W_1$, $W_1 \subseteq W$, and W_1 is also a closed set in X . Equivalently, this means that there is a local base for the topology of X at x consisting of subsets of X that are both open and closed. If X has topological dimension 0 at every point $x \in X$, then X is said to have topological dimension 0 as a topological space. As before, this is the same as saying that there is a base for the topology of X consisting of subsets of X that are both open and closed. Sometimes X would also be required to be nonempty to have topological dimension 0, and the empty set is defined to have topological dimension -1 , in order to define topological dimension inductively when it is positive.

Let X be a topological space with topological dimension 0. Observe that every subset Y of X has topological dimension 0 with respect to the induced topology. Clearly X is regular as a topological space in the strict sense, without including the first or 0th separation condition. One may wish to include the first or 0th separation condition as part of the definition of topological dimension 0, in which case the space is Hausdorff as well. If X has at least two elements, and if X satisfies the first or 0th separation condition, then it is easy to see that X is not connected. It follows that X is totally disconnected when X has topological dimension 0 and X satisfies the first or 0th separation condition, since every subset of X has the same properties with respect to the induced topology. Otherwise, any set equipped with the indiscrete topology has topological dimension 0 in the strict sense, without including the first or 0th separation condition.

Suppose that $d(x, y)$ is a semi-ultrametric on a set X . As in Section 4, both open and closed balls in X with respect to d of positive radius are both open and closed as subsets of X , with respect to the topology on X determined by d . This implies that X has topological dimension 0 in the strict sense, without including the first or 0th separation condition, with respect to the topology determined by d . Similarly, if \mathcal{M} is a nonempty collection of semi-ultrametrics on X , then X has topological dimension 0 in the strict sense with respect to the topology determined by \mathcal{M} . Of course, if \mathcal{M} is nondegenerate on X , then X is Hausdorff with respect to the topology determined by \mathcal{M} .

Let E be a subset of a set X , and put

$$(44.1) \quad d_E(x, y) = 0$$

when $x, y \in E$ and when $x, y \in X \setminus E$, and put

$$(44.2) \quad d_E(x, y) = 1$$

when $x \in E$ and $y \in X \setminus E$, and when $x \in X \setminus E$ and $y \in E$. It is easy to see that this defines a semi-ultrametric on X . More precisely, if E is a nonempty proper subset of X , then

$$(44.3) \quad \{E, X \setminus E\}$$

is a partition of X , and $d_E(x, y)$ is the same as the discrete semimetric associated to this partition, as in the preceding section. Otherwise, if $E = \emptyset$ or $E = X$, then (44.1) holds for every $x, y \in X$, and $d_E(x, y)$ is the same as the discrete semimetric associated to the trivial partition of X . Suppose now that X is a topological space. If $E \subseteq X$ is both open and closed, then d_E is compatible with the topology on X , in the sense that every open set in X with respect to d_E is also an open set with respect to the given topology on X . If

$$(44.4) \quad \mathcal{M}_0 = \{d_E : E \subseteq X \text{ is both open and closed}\},$$

is the collection of all such semi-ultrametrics on X , then it follows that every open set in X with respect to the topology determined by \mathcal{M}_0 is an open set with respect to the given topology on X too. If X has topological dimension 0 in the strict sense, then one can check that the topology on X determined by \mathcal{M}_0 is the same as the given topology on X . If X also satisfies the first or 0th separation condition, then \mathcal{M}_0 is nondegenerate on X .

45 Totally separated topological spaces

A topological space X is said to be *totally separated* if for every pair of points $x, y \in X$ with $x \neq y$ there is an open set $U \subseteq X$ such that $x \in U$, $y \in X \setminus U$, and U is also closed in X . Note that this is symmetric in x and y , since $X \setminus U$ is both open and closed in X too. Suppose that X is totally separated, which implies that X is Hausdorff. It is easy to see that every subset of X is totally separated with respect to the induced topology. If X has at least two elements, then X is not connected. This implies that X is totally disconnected, because subspaces of X are totally separated. If $\tilde{\tau}$ is another topology on X which contains the given topology on X , then X is totally separated with respect to $\tilde{\tau}$ too.

Let (X, τ) be a topological space, and let τ_0 be the collection of subsets W of X with the property that for each $x \in W$ there is an open set $U \subseteq X$ with respect to τ such that $x \in U$, $U \subseteq W$, and U is closed in X with respect to τ . Equivalently, this means that W can be expressed as the union of a family of subsets of X that are both open and closed with respect to τ . In particular, this implies that W is open with respect to τ , so that

$$(45.1) \quad \tau_0 \subseteq \tau.$$

It is easy to see that τ_0 is a topology on X , which is the same as the topology determined by the collection (44.4) of semi-ultrametrics on X associated to τ . By construction, if $U \subseteq X$ is both open and closed with respect to τ , then U is an open set with respect to τ_0 . In this case, $X \setminus U$ is both open and closed with respect to τ too, so that $X \setminus U$ is an open set with respect to τ_0 , which means that U is a closed set with respect to τ_0 . Conversely, if $U \subseteq X$ is both open and closed with respect to τ_0 , then U is both open and closed with respect to τ , because of (45.1). The collection of these sets forms a base for τ_0 , by definition of τ_0 . This implies that X automatically has topological dimension 0

with respect to τ_0 , which also follows from the characterization of τ_0 in terms of (44.4). One can check that X is totally separated with respect to τ if and only if X is Hausdorff with respect to τ_0 .

Let X be a totally separated topological space. Also let H be a compact subset of X , and let $y \in X \setminus H$ be given. If $x \in H$, then $x \neq y$, and so there is an open set $U(x) \subseteq X$ such that $x \in U(x)$, $y \in X \setminus U(x)$, and $U(x)$ is a closed set in X . Because H is compact, there are finitely many elements x_1, \dots, x_n of H such that

$$(45.2) \quad H \subseteq \bigcup_{j=1}^n U(x_j).$$

The right side of (45.2) is both open and closed in X , and does not contain y . Similarly, if H and K are disjoint compact subsets of X , then there is an open set $U \subseteq X$ such that $H \subseteq U$, $K \subseteq X \setminus U$, and U is a closed set in X . If X is a topological space with topological dimension 0, $K \subseteq X$ is compact, $W \subseteq X$ is open, and $K \subseteq W$, then an analogous argument implies that there is an open set $U \subseteq X$ such that $K \subseteq U \subseteq W$ and U is a closed set in X .

Let X be a totally separated topological space again, and suppose that X is locally compact. We would like to check that X has topological dimension 0 under these conditions. To do this, let $x \in X$ and an open set $W \subseteq X$ that contains x be given. Because X is locally compact, there is an open set in X that contains x and is contained in a compact subset of X . We may as well suppose that W is contained in a compact subset of X , since otherwise we can replace W with its intersection with an open set that contains x and is contained in a compact set. Note that compact subsets of X are closed sets, because X is Hausdorff, since it is totally separated. It follows that the closure \overline{W} of W in X is contained in a compact set, and hence that \overline{W} is compact in X , because closed subsets of compact sets are compact. Similarly,

$$(45.3) \quad \partial W = \overline{W} \setminus W$$

is compact, and $x \notin \partial W$, since $x \in W$. As in the previous paragraph, it follows that there is an open set $U_1 \subseteq X$ such that $\partial W \subseteq U_1$, $x \in X \setminus U_1$, and U_1 is a closed set in X , because X is totally separated. Put

$$(45.4) \quad U_2 = U_1 \cup (X \setminus W) = U_1 \cup (X \setminus \overline{W}),$$

where the second step uses the fact that $\partial W \subseteq U_1$. It is easy to see that U_2 is both open and closed in X , because of the analogous property of U_1 , and the two expressions for U_2 in (45.4). Thus $X \setminus U_2$ is both open and closed in X , and $X \setminus U_2 \subseteq W$, since $X \setminus W \subseteq U_2$, by the definition of U_2 . We also have that $x \in X \setminus U_2$, because $x \in X \setminus U_1$, by the way that U_1 was chosen, and $x \in W$, by hypothesis. This shows that X has topological dimension 0 at x , and hence that X has topological dimension 0, since $x \in X$ is arbitrary. It is well known that any locally compact Hausdorff topological space that is totally disconnected has topological dimension 0.

46 Dimension 0, continued

Let A be a commutative topological group. If A has topological dimension 0 at any of its elements, then A has topological dimension 0 at each point, because of continuity of translations. Suppose for the moment that the collection of open subgroups of A is a local base for the topology of A at 0. Equivalently, this means that for open set $W \subseteq A$ that contains 0, there is an open subgroup B of A such that

$$(46.1) \quad B \subseteq W.$$

This implies that A has topological dimension 0 at 0, and hence at every point in A , because open subgroups of A are closed sets, as in Section 42. Note that the intersection of finitely many open subgroups of A is an open subgroup of A too. If there is a local sub-base of A at 0 consisting of open subgroups of A , then it follows that the open subgroups of A form a local base for the topology of A at 0. If the open subgroups of A form a local base for the topology of A at 0, then every subgroup of A has the same property with respect to the induced topology.

Suppose now that for each open set $W \subseteq A$ that contains 0 there are subsets E, U of A containing 0 such that $E \subseteq W$, U is an open set in A , U is symmetric about 0, and E is U -separated from $A \setminus E$, as in Section 41. In particular, this implies that E and $A \setminus E$ are separated in A , as before. It follows that E is both open and closed in A , so that this condition implies that A has topological dimension 0 at 0. Let W, E , and U be given as in this condition, and let B be obtained from U as in (42.6). Thus B is an open subgroup of A , as in Section 42. Because E is U -separated from $A \setminus E$, we also have that $E + B$ is contained in E , as in (42.10). This implies that

$$(46.2) \quad B \subseteq E,$$

since $0 \in E$, and hence that (46.1) holds, because $E \subseteq W$ by hypothesis. This shows that the condition mentioned at the beginning of the paragraph implies that the collection of open subgroups of A is a local base for the topology of A at 0. Of course, if B is any open subgroup of A , then B is B -separated from $A \setminus B$, because $B + B$ is contained in B . If the collection of open subgroups of A is a local base for the topology of A at 0, then it follows that A has the property mentioned at the beginning of the paragraph.

Suppose that A is locally compact at 0, which is to say that there is an open set $W_0 \subseteq A$ that contains 0 and is contained in a compact set $K \subseteq A$. Suppose also that A has topological dimension 0 at 0, and let $W \subseteq A$ be any open set that contains 0. Thus $W \cap W_0$ is an open subset of A that contains 0, and so there is an open set $E \subseteq A$ such that $0 \in E$,

$$(46.3) \quad E \subseteq W \cap W_0,$$

and E is a closed set in A . In particular,

$$(46.4) \quad E \subseteq W_0 \subseteq K,$$

which implies that E is compact in A , because E is a closed set and K is compact. Remember that E is an open set in A too, so that $A \setminus E$ is a closed set. Using an argument from Section 41, we get that there is an open set $U \subseteq A$ such that $0 \in U$, U is symmetric about 0, and E is U -separated from $A \setminus E$. This shows that A satisfies the condition mentioned at the beginning of the previous paragraph under these conditions, so that the collection of open subgroups of A is a local base for the topology of A at 0.

If A is any commutative group equipped with the discrete topology, then $\{0\}$ is an open subgroup of A , which defines a local base for the discrete topology on A . Consider now the set \mathbf{Q} of rational numbers as a commutative topological group with respect to addition and the topology induced by the standard topology on the real line. It is easy to see that \mathbf{Q} has topological dimension 0 as a topological space. However, one can also check that \mathbf{Q} is the only open subgroup of itself.

47 Translation-invariant semi-ultrametrics

Let A be a commutative group, and let B be a subgroup of A . The collection of cosets of B in A defines a partition $\mathcal{P}_B(A)$ of A . Let

$$(47.1) \quad d_{\mathcal{P}_B(A)}(x, y)$$

be the discrete semi-ultrametric on A corresponding to $\mathcal{P}_B(A)$ as in Section 43. In this case, (47.1) is equal to 0 when

$$(47.2) \quad x - y \in B,$$

and otherwise (47.1) is equal to 1. In particular, (47.1) is invariant under translations on A . Note that $\mathcal{P}_B(A)$ corresponds to the standard quotient mapping from A onto the quotient group A/B as in (43.3). Thus (47.1) corresponds to the discrete metric on A/B and the standard quotient mapping from A onto A/B as in (43.2).

If $d(x, y)$ is a q -semimetric on A for some $q > 0$, then we have seen that (43.4) defines an equivalence relation on A , for which the corresponding equivalence classes are the same as the closed balls in A of radius 0. If $d(x, y)$ is invariant under translations on A , then one can check that the equivalence class containing 0 is a subgroup of A , and the other equivalence classes are cosets of this subgroup in A . Similarly, if $d(x, y)$ is a semi-ultrametric on A and $r > 0$, then A is partitioned by the open balls of radius r with respect to $d(x, y)$, and by the closed balls of radius r . If $d(x, y)$ is invariant under translations, then one can verify that the open and closed balls in A with respect to $d(x, y)$ centered at 0 with radius r are subgroups of A for every $r > 0$. Of course, the open and closed balls in A with respect to $d(x, y)$ centered at other points are cosets of the corresponding balls centered at 0.

Let \mathcal{M} be a nonempty collection of translation-invariant semi-ultrametrics on A , and remember that A is a commutative topological group with respect to

the topology determined by \mathcal{M} , as in Section 19. As in the preceding paragraph, the open and closed balls in A centered at 0 with respect to elements of \mathcal{M} are subgroups of A . Of course, open balls in A with respect to elements of \mathcal{M} are open sets with respect to the corresponding topology. In this situation, closed balls in A of positive radius with respect to elements of \mathcal{M} are open sets too, because the elements of \mathcal{M} are semi-ultrametrics, as in Section 4. Remember that open balls in A centered at 0 with respect to \mathcal{M} form a local sub-base for the topology on A determined by \mathcal{M} , as in Section 2. This is a local sub-base for the topology of A at 0 consisting of open subgroups of A , since open balls in A centered at 0 with respect to elements of \mathcal{M} are subgroups of A , as before. It follows that the open subgroups of A form a local base for the topology of A at 0 under these conditions, as in the previous section.

Let \mathcal{B} be a nonempty collection of subgroups of A , and let

$$(47.3) \quad \mathcal{M}(\mathcal{B}) = \{d_{\mathcal{P}_B(A)} : B \in \mathcal{B}\}$$

be the collection of discrete semi-ultrametrics on A corresponding to the partitions $\mathcal{P}_B(A)$ of A associated to elements B of \mathcal{B} as in (47.1). Remember that these semi-ultrametrics are invariant under translations on A . This implies that A is a commutative topological group with respect to the topology determined by $\mathcal{M}(\mathcal{B})$, and the open subgroups of A form a local base for the topology of A at 0, as in the preceding paragraph. More precisely, each $B \in \mathcal{B}$ is the same as the open ball in A centered at 0 with radius 1 with respect to the corresponding discrete semi-ultrametric $d_{\mathcal{P}_B(A)}$, by construction. Using this, it is easy to see that \mathcal{B} is a local sub-base for the topology of A at 0 with respect to the topology determined by $\mathcal{M}(\mathcal{B})$.

Let A be a commutative topological group, and put

$$(47.4) \quad \mathcal{B}(A) = \{B \subseteq A : B \text{ is an open subgroup of } A\}.$$

Note that A is automatically an element of $\mathcal{B}(A)$, so that $\mathcal{B}(A) \neq \emptyset$. Also let $\mathcal{M}(\mathcal{B}(A))$ be the collection of discrete semi-metrics corresponding to the partitions $\mathcal{P}_B(A)$ associated to $B \in \mathcal{B}(A)$, as in (47.3). If $B \in \mathcal{B}(A)$, then the corresponding discrete semi-metric $d_{\mathcal{P}_B(A)}$ is compatible with the given topology on A , in the sense that every open set in A with respect to $d_{\mathcal{P}_B(A)}$ is also an open set with respect to the given topology on A . This implies that every open set in A with respect to the topology determined by $\mathcal{M}(\mathcal{B}(A))$ is an open set with respect to the given topology on A . Of course, every element of $\mathcal{B}(A)$ is an open set in A with respect to the topology determined by $\mathcal{M}(\mathcal{B}(A))$. If $\mathcal{B}(A)$ is a local base for the given topology on A at 0, then the topology on A determined by $\mathcal{M}(\mathcal{B}(A))$ is the same as the given topology on A .

Suppose that for each $y \in A$ with $y \neq 0$ there is an open set $U \subseteq A$ such that $0 \in U$, $y \in A \setminus U$, and U is a closed set in A too. This is the same as the totally separated property described in Section 45 with $x = 0$. In this situation, this property implies that A is totally separated as a topological space, because of continuity of translations.

Suppose now that for each $y \in A$ with $y \neq 0$ there is an open subgroup B of A such that $y \notin B$. This implies the condition mentioned in the previous paragraph, because open subgroups of A are closed sets in A , as in Section 42. Equivalently, this new condition is the same as saying that the intersection of all of the open subgroups of A is equal to $\{0\}$. This is also equivalent to asking that $\mathcal{M}(\mathcal{B}(A))$ be nondegenerate on A . If A has this property, then every subgroup of A has the same property with respect to the induced topology.

48 Cartesian products, continued

Let I be a nonempty set, and let Y_j be a topological space for each $j \in I$. Thus the Cartesian product

$$(48.1) \quad Y = \prod_{j \in I} Y_j$$

is also a topological space with respect to the corresponding product and strong product topologies. If $E_j \subseteq Y_j$ is a closed set for each $j \in I$, then

$$(48.2) \quad E = \prod_{j \in I} E_j$$

is a closed set in Y with respect to the product topology, and hence with respect to the strong product topology. If $U_j \subseteq Y_j$ is an open set for each $j \in I$, and if $U_j = Y_j$ for all but finitely many $j \in I$, then

$$(48.3) \quad U = \prod_{j \in I} U_j$$

is an open set in Y with respect to the product topology. If U_j is a closed set in Y_j for each $j \in I$ too, then U is a closed set in Y with respect to the product topology, as before. Using this, it is easy to see that if Y_j has topological dimension 0 for each j , then Y has topological dimension 0 with respect to the product topology. Similarly, if Y_j is totally separated for each $j \in I$, then Y is totally separated with respect to the product topology.

If $U_j \subseteq Y_j$ is an open set for each $j \in I$, then (48.3) is an open set in Y with respect to the strong product topology on Y . If U_j is also a closed set in Y_j for each $j \in I$, then U is a closed set in Y with respect to the strong product topology, as in the preceding paragraph. If Y_j has topological dimension 0 for each $j \in I$, then it follows that Y has topological dimension 0 with respect to the strong product topology too, as before. If Y_j is totally separated for each $j \in I$, then Y is totally separated with respect to the strong product topology as well. However, this follows from the analogous statement for the product topology on Y , since open and closed subsets of Y with respect to the product topology have the same property with respect to the strong product topology.

Let X be another topological space, and let ϕ_j be a continuous mapping from X into Y_j for each $j \in I$. This leads to a mapping

$$(48.4) \quad \phi : X \rightarrow Y$$

in a natural way, where the j th component of $\phi(x)$ in Y is equal to $\phi_j(x)$ for each $x \in X$ and $j \in I$. It is easy to see that ϕ is continuous with respect to the product topology on Y , but this does not always work with respect to the strong product topology on Y . Suppose now that

$$(48.5) \quad Y_j = \{0, 1\}$$

equipped with the discrete topology for each $j \in I$, so that Y has topological dimension 0 with respect to the corresponding product topology, as before. In this case, ϕ_j is continuous if and only if $\phi_j^{-1}(\{0\})$ and $\phi_j^{-1}(\{1\})$ are open subsets of X , which is the same as saying that $\phi_j^{-1}(\{1\})$ is both open and closed in X .

Let W_j be a subset of X that is both open and closed for each $j \in I$, and let ϕ_j be the mapping from X into $\{0, 1\}$ such that

$$(48.6) \quad \phi_j = 1 \text{ on } W_j \quad \text{and} \quad \phi_j = 0 \text{ on } X \setminus W_j$$

for each $j \in I$. Thus ϕ_j is continuous for each $j \in I$, so that the corresponding mapping ϕ from X into the Cartesian product Y is continuous with respect to the product topology on Y , as in the previous paragraph. Suppose that for each $x, x' \in X$ with $x \neq x'$ there is a $j \in I$ such that

$$(48.7) \quad x \in W_j \text{ and } x' \notin W_j, \quad \text{or} \quad x' \in W_j \text{ and } x \notin W_j.$$

This is the same as saying that the corresponding mapping ϕ from X into Y is injective. Note that X is totally separated if and only if there is a family of subsets of X that are both open and closed, and which separates points in X in this way. If, in addition to these conditions, $\{W_j\}_{j \in I}$ is a sub-base for the topology of X , then ϕ is a homeomorphism from X onto its image in Y , with respect to the topology induced on $\phi(X)$ by the product topology on Y . Remember that X has topological dimension 0 if and only if there is a base for the topology of X consisting of subsets of X that are both open and closed, as in Section 44. If there is a sub-base for the topology of X consisting of subsets of X that are both open and closed, then the corresponding base for the topology of X consisting of finite intersections of elements of the sub-base has the same property. If X also satisfies the first or 0th separation condition, then any sub-base for the topology of X separates points as well.

49 Direct products

Let I be a nonempty set, and let C_j be a commutative topological group for each $j \in I$. As in Section 23, the Cartesian product

$$(49.1) \quad C = \prod_{j \in I} C_j$$

is a commutative group, where the group operations are defined coordinatewise. We have also seen that C is a commutative topological group with respect to

the corresponding product topology. Let D_j be a subgroup of C_j for each $j \in I$, so that

$$(49.2) \quad D = \prod_{j \in I} D_j$$

is a subgroup of C . If D_j is an open set in C_j for each j , and if $D_j = C_j$ for all but finitely many j , then D is an open set in C with respect to the product topology. If the open subgroups of C_j form a local base for the topology of C_j at 0 for every $j \in I$, then one can check that the open subgroups of C form a local base for the product topology on C at 0. If, for each $j \in I$, the intersection of the open subgroups of C_j is equal to $\{0\}$, then C has the analogous property with respect to the product topology.

Similarly, C is a commutative topological group with respect to the strong product topology, as in Section 31. If D_j is an open subgroup of C_j for each $j \in I$, then (49.2) is an open subgroup of C with respect to the strong product topology. As in the preceding paragraph, if, for each $j \in I$, the intersection of the open subgroups of C_j is equal to $\{0\}$, then C has the same property with respect to the strong product topology. This follows from the analogous statement for the product topology on C , since open subsets of C with respect to the product topology are open with respect to the strong product topology as well.

Let A be another commutative topological group, and let ϕ_j be a continuous homomorphism from A into C_j for each $j \in I$. As in the previous section, this leads to a mapping

$$(49.3) \quad \phi : A \rightarrow C,$$

whose j th component is equal to ϕ_j for each $j \in I$. Under these conditions, ϕ is a continuous homomorphism from A into C , with respect to the product topology on C . If C_j is equipped with the discrete topology for each $j \in I$, then the open subgroups of C with respect to the product topology form a local base for C at 0, as mentioned earlier. In this situation, ϕ_j is continuous if and only if the kernel $\phi_j^{-1}(\{0\})$ of ϕ_j is an open subgroup of A .

Let B_j be an open subgroup of A for each $j \in I$, and let

$$(49.4) \quad C_j = A/B_j$$

be the quotient of A by B_j , equipped with the discrete topology. Also let ϕ_j be the corresponding quotient mapping from A onto C_j for each $j \in I$, so that

$$(49.5) \quad \phi_j^{-1}(\{0\}) = B_j$$

for each $j \in I$. Thus ϕ_j is continuous for each $j \in I$, as in the preceding paragraph. This implies that the corresponding mapping ϕ from A into the product C of the C_j 's is a continuous homomorphism with respect to the product topology on C , as before. By construction,

$$(49.6) \quad \phi^{-1}(\{0\}) = \bigcap_{j \in I} \phi_j^{-1}(\{0\}) = \bigcap_{j \in I} B_j,$$

so that ϕ is injective if and only if

$$(49.7) \quad \bigcap_{j \in I} B_j = \{0\}.$$

If (49.7) holds, and if $\{B_j\}_{j \in I}$ is a local sub-base for the topology of A at 0, then ϕ is a homeomorphism from A onto its image in C , with respect to the product topology on C . Of course, if $\{0\}$ is a closed set in A , then $\{x\}$ is a closed set in A for every $x \in A$, which implies that the intersection of all open subsets of A that contain 0 is equal to $\{0\}$. In this case, if $\{B_j\}_{j \in I}$ is a local sub-base for the topology of A at 0, then (49.7) holds automatically.

50 Weak connectedness

Let us say that a commutative topological group A is *weakly connected* if there are no proper open subgroups of A . If A is connected as a topological space in the usual sense, then A is weakly connected, because open subgroups of A are closed sets in A , as in Section 42. Remember that the set \mathbf{Q} of rational numbers has topological dimension 0 with respect to the topology induced by the standard topology on \mathbf{R} , and in particular \mathbf{Q} is totally disconnected. We have also seen that \mathbf{Q} is weakly connected as a commutative topological group with respect to addition and this topology, as in Section 46.

Suppose that A_1, A_2 are commutative topological groups, and that ϕ is a continuous homomorphism from A_1 into A_2 . If B_2 is an open subgroup of A_2 , then $\phi^{-1}(B_2)$ is an open subgroup of A_1 . If A_1 is weakly connected, then

$$(50.1) \quad \phi^{-1}(B_2) = A_1,$$

which is to say that

$$(50.2) \quad \phi(A_1) \subseteq B_2.$$

As before, B_2 is a closed set in A_2 , so that (50.2) implies that

$$(50.3) \quad \overline{\phi(A_1)} \subseteq B_2,$$

where $\overline{\phi(A_1)}$ is the closure of $\phi(A_1)$ in A_2 . If $\phi(A_1)$ is dense in A_2 , then it follows that A_2 is weakly connected as well.

Let us say that a subgroup A_0 of a commutative topological group A is weakly connected if A_0 is weakly connected as a commutative topological group, with respect to the topology induced by the one on A . If A_0 is a weakly connected subgroup of A and B is an open subgroup of A , then

$$(50.4) \quad A_0 \subseteq B.$$

This follows from (50.2), applied to the standard inclusion mapping of A_0 into A as a continuous homomorphism, or by observing that $A_0 \cap B$ is a relatively open subgroup of A_0 . Since (50.4) holds for every weakly connected subgroup A_0 of A ,

we get that the subgroup of A generated by all of its weakly connected subgroups is contained in B . If A is generated by its weakly connected subgroups, then A is weakly connected too.

Let A_0 be a subgroup of a commutative topological group A again, and let B_0 be a subgroup of A_0 . Also let

$$(50.5) \quad B = \overline{B_0}$$

be the closure of B_0 in A , which is a closed subgroup of A . Suppose that B_0 is a relatively open subgroup of A_0 , so that B_0 is relatively closed in A_0 too, as in Section 42. This implies that

$$(50.6) \quad B \cap A_0 = B_0.$$

Because B_0 is relatively open in A , there is an open set $U \subseteq A$ such that

$$(50.7) \quad B_0 = U \cap A_0.$$

If A_0 is a dense subset of A , then we get that

$$(50.8) \quad U \subseteq \overline{U \cap A_0} = \overline{B_0} = B.$$

This implies that B is an open set in A , using continuity of translations on A , and the fact that $0 \in B_0 \subseteq U$. Note that $B \neq A$ when $B_0 \neq A_0$, by (50.6). It follows that A_0 is weakly connected when A is weakly connected and A_0 is dense in A .

51 Weak connectedness, continued

Let V be a topological vector space over the field \mathbf{Q} of rational numbers, with respect to the standard absolute value function on \mathbf{Q} . Suppose that B is an open subset of V which is also a subgroup of V as a commutative group with respect to addition. As in Section 20,

$$(51.1) \quad \phi_v(t) = tv$$

defines a continuous linear mapping from \mathbf{Q} into V for every $v \in V$. This implies that $\phi_v^{-1}(B)$ is an open subgroup of \mathbf{Q} with respect to addition for every $v \in V$. It follows that

$$(51.2) \quad \phi_v^{-1}(B) = \mathbf{Q}$$

for every $v \in V$, because \mathbf{Q} is weakly connected as a commutative topological group. In particular,

$$(51.3) \quad v = \phi_v(1) \in B$$

for every $v \in V$. This shows that

$$(51.4) \quad B = V,$$

so that V is weakly connected as a commutative topological group with respect to addition under these conditions.

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Suppose that $|\cdot|$ is archimedean on k , as in Sections 5 and 8. This implies that k has characteristic 0, so that there is a natural embedding of \mathbf{Q} into k . Thus $|\cdot|$ induces a q -absolute value function on \mathbf{Q} , which is archimedean as well. Using Ostrowski's theorem, as in Section 8, we get that this induced q -absolute value function on \mathbf{Q} is equivalent to the standard absolute value function, which means that the induced absolute value function on \mathbf{Q} is equal to a positive power of the standard absolute value function on \mathbf{Q} .

Let V be a topological vector space over k . We can also think of V as a topological vector space over \mathbf{Q} , using the natural embedding of \mathbf{Q} in k , and the q -absolute value function on \mathbf{Q} induced by $|\cdot|$ on k . Because the induced absolute value function on \mathbf{Q} is equivalent to the standard absolute value function on \mathbf{Q} , as in the preceding paragraph, we can think of V as a topological vector space over \mathbf{Q} with respect to the standard absolute value function on \mathbf{Q} too. By the remarks at the beginning of the section, V is weakly connected as a commutative topological group with respect to addition.

Of course, if $k = \mathbf{R}$ or \mathbf{C} with the standard absolute value function, then k is connected with respect to the corresponding topology. Similarly, if V is a topological vector space over \mathbf{R} or \mathbf{C} , with respect to the standard absolute value function, then V is connected, and in fact pathwise connected. In particular, this implies that V is weakly connected as a commutative topological group with respect to addition, as in the previous section.

52 Open subgroups, continued

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k . If E_1, E_2 are balanced subsets of V , then it is easy to see that their sum

$$(52.1) \quad E_1 + E_2$$

is balanced in V too. Suppose that $U \subseteq V$ contains 0 and is symmetric about 0, and let $B \subseteq V$ be as in (42.6). Equivalently, if we consider V as a commutative group with respect to addition, then B is the subgroup of V generated by U . If U is also balanced in V , then one can check that B is balanced in V too. More precisely, if U_j is as in Section 42 for each $j \in \mathbf{Z}_+$, then one can verify that U_j is balanced for every j , using induction on j . This implies that B is balanced, because B is defined as the union of the U_j 's. Suppose now that V is a topological vector space over k , so that V is a commutative topological group with respect to addition in particular. If $U \subseteq V$ is an open set that contains 0 and is symmetric about 0, and if B is as in (42.6) again, then B is an open subgroup of V , as in Section 42. If $|\cdot|$ is archimedean on k , then it follows that $B = V$, as in the previous section.

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let V be a topological vector space over k . Suppose that $B_0 \subseteq V$ is an open set which is

also a subgroup of V as a commutative group with respect to addition. Suppose too that there is a nonempty balanced open set $U \subseteq V$ that is contained in B_0 . In particular, this means that U is symmetric about 0. Let B be the subgroup of V generated by U as in (42.6). Thus B is a balanced open subset of V under these conditions, as in the previous paragraphs. We also have that

$$(52.2) \quad B \subseteq B_0,$$

because $U \subseteq B_0$ and B_0 is a subgroup of V with respect to addition. If $|\cdot|$ is nontrivial on k , then there is always a nonempty balanced open set $U \subseteq B_0$, as in Section 20. Similarly, if $|\cdot|$ is the trivial absolute value function on k , and if the collection of linear mappings on V corresponding to multiplication by elements of k is equicontinuous at 0, then there is always a nonempty balanced open set $U \subseteq B_0$.

Let k, V be as in the preceding paragraph, and suppose for the moment that $|\cdot|$ is nontrivial on k . Also let B be a subgroup of V with respect to addition that is balanced and absorbing in V . In particular, if B is an open set in V , then B is absorbing, as in Section 20. Note that B is ∞ -convex in V , in the sense described in Section 13, because B is a balanced subgroup of V . If N_B is the Minkowski functional on V corresponding to B as in (14.1), then N_B is a semi-ultranorm on V under these conditions, as in Section 14.

Suppose now that $|\cdot|$ is the trivial absolute value function on k . In this case, if B is a subgroup of V with respect to addition that is balanced, then B is a linear subspace of V . If we put

$$(52.3) \quad N(v) = 0 \text{ when } v \in B \quad \text{and} \quad N(v) = 1 \text{ when } v \in V \setminus B,$$

then it is easy to see that N is a semi-ultranorm on V . This is the same as the composition of the standard quotient mapping from V onto the quotient vector space V/B with the trivial ultranorm on V/B . The semi-ultrametric on V associated to N as in (6.3) is the same as the discrete semimetric on V corresponding to the partition of V into cosets of B , as in Sections 43 and 47.

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